Optimization Modulo Theories
An Introduction

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Outline

1. Motivations
2. Optimization Modulo Theories with Linear-Arithmetic Objectives
3. OMT with Multiple and Combined Objectives
4. Relevant Subcases: OMT+PB & MaxSMT
5. Status of OMT
6. Current and Future Research Directions
7. Appendix
   - Inline OMT schema
   - OMT for Bit-vector and Floating-point theories
   - Imptoving OMT+PB by sorting networks
   - The MaxRES MaxSMT Procedure
   - Extended SMT-LIB language
   - Pareto Optimization (hints)

Outline

Satisfiability Modulo Theories SMT($\mathcal{T}$)

SMT($\mathcal{T}$): the problem of deciding the satisfiability of a (typically) ground first-order formula wrt some background theory $\mathcal{T}$.

- $\mathcal{T}$ can be a combination of theories $\bigcup_i \mathcal{T}_i$
- Theories of Interest:
  - Linear arithmetic over the rationals ($\mathcal{LRA}$)
    $(T_\delta \rightarrow (s_1 = s_0 + 3.4 \cdot t - 3.4 \cdot t_0)) \wedge (\neg T_\delta \rightarrow (s_1 = s_0))$
  - Linear arithmetic over the integers ($\mathcal{LIA}$)
    $(x := x_l + 2^{16} x_h) \wedge (x \geq 0) \wedge (x \leq 2^{16} - 1)$
  - Arrays ($\mathcal{AR}$)
    $(i = j) \vee read(write(a, i, e), j) = read(a, j)$
  - Bit vectors ($\mathcal{BV}$)
    $x_{[16]}[15:0] = (y_{[16]}[15:8] :: z_{[16]}[7:0]) << w_{[8]}[3:0]$
  - Non-linear arithmetic ($\mathcal{NLA}$)
    $((c = a \cdot b) \wedge (a_1 = a - 1) \wedge (b_1 = a + 1)) \rightarrow (c = a_1 \cdot b_1 + 1)$
  - ...
- "Lazy" Approach: SMT solver = CDCL SAT solver + $\mathcal{T}$-solver(s)

Need for Satisfiability Modulo Theories (SMT)

SMT solvers widely used as backend engines in formal verification and many other applications

- SW verification
- verification of Timed and Hybrid Systems
- verification of RTL Circuit designs & of microcode
- static analysis of SW programs
- test-case generation
- program synthesis
- scheduling
- planning with resources
- compiler optimization
- ...

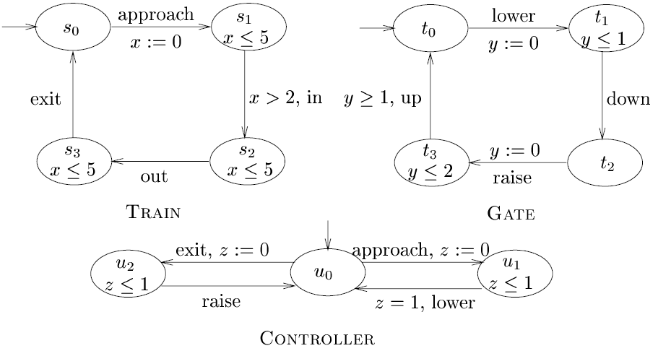
Need for Optimization Modulo Theories (SMT)

Many SMT-encodable problems require optimum solutions wrt. some objective function. E.g.:

- SW verification
- formal verification of parametric systems
- optimization of physical layout of circuit designs
- scheduling and temporal reasoning
- displacement of tools (e.g. strip-packing problem)
- planning with resources and retrofit planning
- radio link frequency assignment
- machine learning on hybrid domains
- goal modeling in requirement engineering
- ...

Ex.: FV of parametric systems

A (parametric version of a) timed system from [Alur, CAV-99] [8]:



Decision Problem: check safety under fixed choices of the constants (e.g, the delay after which the controller orders the gate to lower the bar) ($M \models \mathbf{G}\neg(in \wedge up)$)

- BMC encodable into a SMT($\mathcal{LRA}$) problem (sat. $\Longrightarrow$ unsafe)

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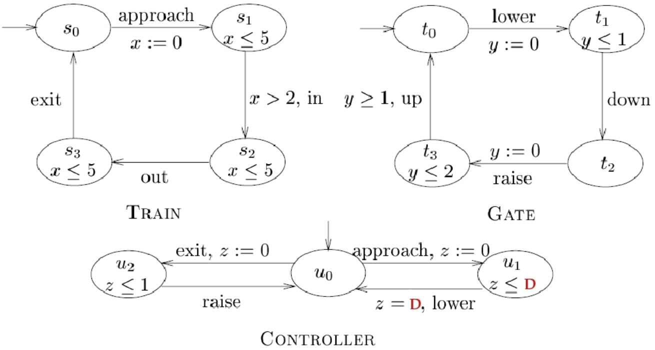

Optimization Problem: find the minimum "unsafe" delay D after which the controller orders the gate to lower the bar, which doesn't guarantee safety ($M \not\models \mathbf{G}\neg(in \wedge up)$).

$\Longrightarrow$ Set the delay D strictly smaller

- BMC encodable into a OMT($\mathcal{LRA}$) problem (min. D s.t. satisf.)

Ex.: Formal Verification of Real-Time Systems

Model Checking: $M \models f$?

Bounded Model Checking (BMC) looks for an execution path of M of (increasing) length k

- satisfying the temporal property $\neg f$ (i.e. $M \models_k E\neg f$)
- minimizing the total elapsed time: $cost = \min(t^N - t^0)$

BMC is encoded into SMT($\mathcal{T}$) (e.g. $\mathcal{T} = \mathcal{LRA} \cup \mathcal{AR} \cup \ldots$):

- if φ_k is satisfiable, then $M \not\models f$

$$
\begin{array}{lccl}
 & DUMP^1 & \rightarrow & (A^1 = write(A^0, i^1, v_i^1)) \\
\wedge & \neg DUMP^1 & \rightarrow & (A^1 = A^0) \\
\wedge & DUMP^1 & \rightarrow & (t^1 - t^0 = 0) \\
\wedge & \ldots & & \\
\wedge & WAIT^1 & \rightarrow & (t^1 - t^0 > 0) \\
\wedge & \ldots & & \\
\wedge & DUMP^N & \rightarrow & \ldots \\
\wedge & \ldots & &
\end{array}
$$

Ex.: Formal Verification of Real-Time Systems

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\wedge & \ldots & & \\
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\wedge & \ldots & & \\
\wedge & DUMP^N & \rightarrow & \ldots \\
\wedge & \ldots & &
\end{array}
$$

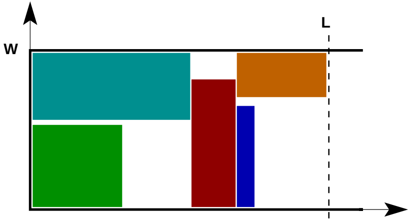
Ex.: Planning with Resources [62]

- SAT-based planning augmented with numerical constraints
- Straightforward to encode into into SMT($\mathcal{LRA}$)
- Goal: find a plan minimizing some resource consumption (time, money, gasoline, ...)

Example (sketch) [62]

$$
\begin{array}{ll}
(\mathit{Deliver}) & \wedge \;\texttt{// goal} \\
(\mathit{MaxLoad}) & \wedge \;\texttt{// load constraint} \\
(\mathit{MaxFuel}) & \wedge \;\texttt{// fuel constraint} \\
(\mathit{Move} \rightarrow \mathit{MinFuel}) & \wedge \;\texttt{// move requires fuel} \\
(\mathit{Move} \rightarrow \mathit{Deliver}) & \wedge \;\texttt{// move implies delivery} \\
(\mathit{GoodTrip} \rightarrow \mathit{Deliver}) & \wedge \;\texttt{// a good trip requires} \\
(\mathit{GoodTrip} \rightarrow \mathit{AllLoaded}) & \wedge \;\texttt{// a full delivery} \\
\hline
(\mathit{MaxLoad} \rightarrow (\mathit{load} \leq 30)) & \wedge \;\texttt{// load limit} \\
(\mathit{MaxFuel} \rightarrow (\mathit{fuel} \leq 15)) & \wedge \;\texttt{// fuel limit} \\
(\mathit{MinFuel} \rightarrow (\mathit{fuel} \geq 7 + 0.5\mathit{load})) & \wedge \;\texttt{// fuel constraint} \\
(\mathit{AllLoaded} \rightarrow (\mathit{load} = 45)) & \quad\;\texttt{//}
\end{array}
$$

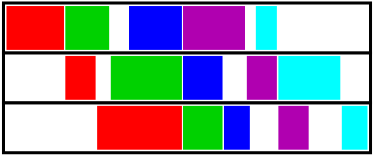
Ex.: (LGDP/MILP) Strip-packing & Carpet-cutting [29, 51, 53]



Strip-packing: Minimize the length L of a strip of width W while fitting N rectangles (no overlap, no rotation) [29]. Carpet-cutting: w. rotation.

$$
\begin{array}{rcl}
\varphi & \stackrel{\text{def}}{=} & (\text{cost} = L) \wedge \bigwedge_{i \in N}(L \geq x_i + L_i) \\
& \wedge & \bigwedge_{i,j \in N, i<j} \Big((x_i + L_i \leq x_j) \vee (x_j + L_j \leq x_i) \\
& & \vee (y_i - H_i \geq y_j) \vee (y_j - H_j \geq y_i) \Big) \\
& \wedge & \bigwedge_{i \in N}(x_i \leq \text{ub} - L_i) \wedge \bigwedge_{i \in N}(x_i \geq 0) \\
& \wedge & \bigwedge_{i \in N}(H_i \leq y_i) \wedge \bigwedge_{i \in N}(W \geq y_i) \wedge \bigwedge_{i \in N}(y_i \geq 0)
\end{array}
$$

Ex.: (LGDP/MILP) Zero-Wait Jobshop Scheduling [29, 51, 53]



Given a set I of jobs which must be scheduled sequentially on a set J of consecutive stages with zero-wait transfer between them, minimize the makespan M [47].

$$
\begin{aligned}
\varphi \quad \stackrel{\text{def}}{=} \quad & (\text{cost} = M) \wedge \bigwedge_{i \in I} (M \geq s_i + \textstyle\sum_{j \in J_i} t_{ij}) \wedge \bigwedge_{i \in I} (s_i \geq 0) \\
\wedge \quad & \bigwedge_{j \in C_{ik}, i,k \in I, i<k} \Big((s_i + \textstyle\sum_{m \in J_i, m \leq j} t_{im} \leq s_k + \sum_{m \in J_k, m<j} t_{km}) \\
& \qquad \vee (s_k + \textstyle\sum_{m \in J_k, m \leq j} t_{km} \leq s_i + \sum_{m \in J_i, m<j} t_{im}) \Big)
\end{aligned}
$$

Outline

Optimization Modulo Theories: General Case

Ingredients

- a SMT formula φ in some background theory $\mathcal{T} = \mathcal{T}_{\preceq} \cup \bigcup_i \mathcal{T}_i$
  - $\bigcup_i \mathcal{T}_i$ may be empty
  - $\mathcal{T}_{\preceq}$ has a predicate $\preceq$ representing a total order
- a $\mathcal{T}_{\preceq}$-variable/term "cost" occurring in φ

Optimization Modulo $\mathcal{T}_{\preceq} \cup \bigcup_i \mathcal{T}_i$ (OMT($\mathcal{T}_{\preceq} \cup \bigcup_i \mathcal{T}_i$))

The problem of finding a model $\mathcal{M}$ for φ whose value of cost is minimum according to $\preceq$.

- maximization dual

Optimization Modulo Theories with $\mathcal{LIRA}$ costs

Ingredients

- an SMT formula φ on $\mathcal{LIRA} \cup \mathcal{T}$
  - $\mathcal{LIRA}$ can be $\mathcal{LRA}$, $\mathcal{LIA}$ or a combination of both
  - $\mathcal{T} \stackrel{\text{def}}{=} \bigcup_i \mathcal{T}_i$, possibly empty
  - $\mathcal{LIRA}$ and $\mathcal{T}_i$ disjoint Nelson-Oppen theories
- a $\mathcal{LIRA}$ variable [term] "cost" occurring in φ
- (optionally) two constant numbers lb (lower bound) and ub (upper bound) s.t. lb $\leq$ *cost* $<$ ub (lb, ub may be $\mp\infty$)

Optimization Modulo Theories with $\mathcal{LIRA}$ costs (OMT($\mathcal{LIRA} \cup \mathcal{T}$))

Find a model for φ whose value of cost is minimum.

- maximization dual

We first restrict to the case $\mathcal{LIRA} = \mathcal{LRA}$ and $\bigcup_i \mathcal{T}_i = \{\}$ (OMT($\mathcal{LRA}$)).

Optimization Modulo Theories with $\mathcal{LRA}$ costs

Ingredients

- an SMT formula φ on $\mathcal{LRA} \cup \mathcal{T}$
  - $\mathcal{LIRA}$ can be $\mathcal{LRA}$, $\mathcal{LIA}$ or a combination of both
  - $\mathcal{T} \stackrel{\text{def}}{=} \bigcup_i \mathcal{T}_i$, possibly empty
  - $\mathcal{LRA}$ and $\mathcal{T}_i$ disjoint Nelson-Oppen theories
- a $\mathcal{LRA}$ variable [term] "cost" occurring in φ
- (optionally) two constant numbers lb (lower bound) and ub (upper bound) s.t. lb $\leq$ *cost* $<$ ub (lb, ub may be $\mp\infty$)

Optimization Modulo Theories with $\mathcal{LRA}$ costs (OMT($\mathcal{LRA} \cup \mathcal{T}$))

Find a model for φ whose value of cost is minimum.

- maximization dual

We first restrict to the case $\mathcal{LIRA} = \mathcal{LRA}$ and $\bigcup_i \mathcal{T}_i = \{\}$ (OMT($\mathcal{LRA}$)).

Solving OMT($\mathcal{LRA}$) [52, 53]

General idea

Combine standard SMT and LP minimization techniques.

Offline Schema

- Minimizer: based on the Simplex $\mathcal{LRA}$-solver by [25]
  - Handles strict inequalities
- Search Strategies:
  - Linear-Search strategy
  - Mixed Linear/Binary strategy

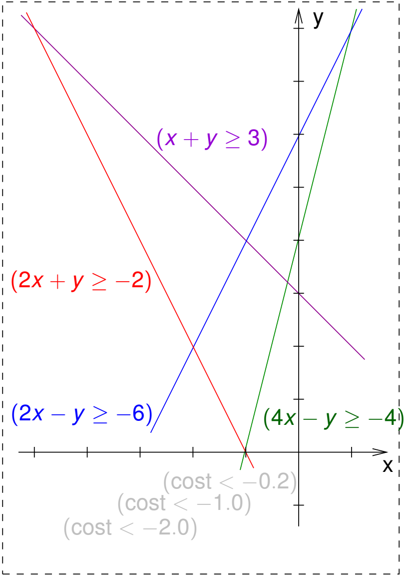
A toy example (linear search)

[w. pure-literal filt. $\Longrightarrow$ partial assignments]

- OMT($\mathcal{LRA}$) problem:

$$
\begin{array}{rl}
\varphi \stackrel{\text{def}}{=} & (\neg A_1 \vee (2x + y \geq -2)) \\
\wedge & (\;\; A_1 \vee (x + y \geq 3)) \\
\wedge & (\neg A_2 \vee (4x - y \geq -4)) \\
\wedge & (\;\; A_2 \vee (2x - y \geq -6)) \\
\wedge & (\text{cost} < -0.2) \\
\wedge & (\text{cost} < -1.0) \\
\wedge & (\text{cost} < -2.0) \\
\text{cost} \stackrel{\text{def}}{=} & x
\end{array}
$$

- $\mu = \left\{ \begin{array}{l} A_1, \neg A_1, \quad A_2, \neg A_2, \\ (4x - y \geq -4), \\ (x + y \geq 3), \\ (2x + y \geq -2), \\ (2x - y \geq -6) \\ (\text{cost} < -0.2) \\ (\text{cost} < -1.0) \\ (\text{cost} < -2.0) \end{array} \right\}$



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\wedge & (\text{cost} < -0.2) \\
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$$\text{cost} \stackrel{\text{def}}{=} x$$

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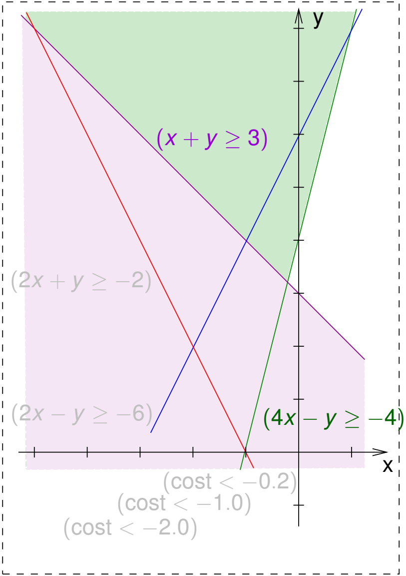
$\Longrightarrow$ SAT, $min = -0.2$



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\wedge & (cost < -0.2) \\
\wedge & (cost < -1.0) \\
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\end{array}
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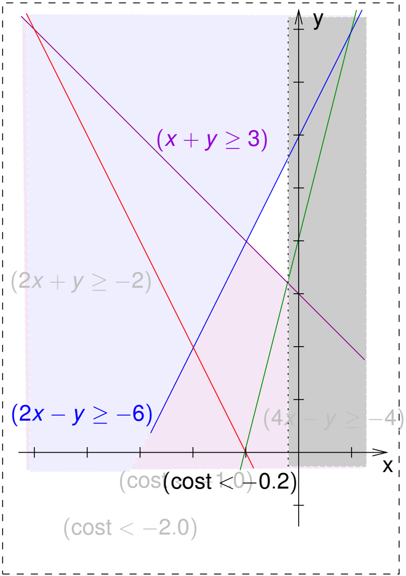
$$cost \stackrel{\text{def}}{=} x$$

- $\mu = \left\{ \begin{array}{l} A_1, \neg A_1, \ \ A_2, \neg A_2, \\ (4x - y \geq -4), \\ (x + y \geq 3), \\ (2x + y \geq -2), \\ (2x - y \geq -6) \\ (cost < -0.2) \\ (cost < -1.0) \\ (cost < -2.0) \end{array} \right\}$

$\Longrightarrow$ SAT, $min = -1.0$



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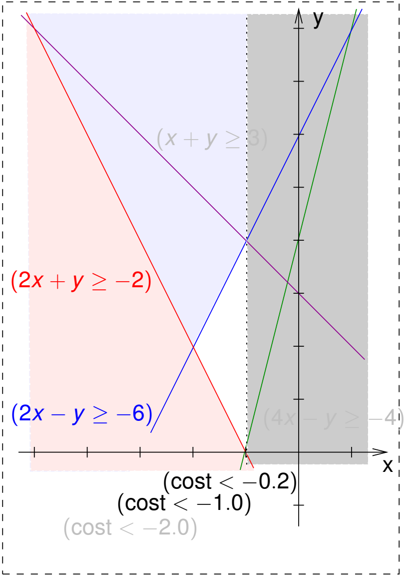
$$\text{cost} \stackrel{\text{def}}{=} x$$

- $$\mu = \left\{ \begin{array}{l} A_1, \neg A_1, \; A_2, \neg A_2, \\ (4x - y \geq -4), \\ (x + y \geq 3), \\ (2x + y \geq -2), \\ (2x - y \geq -6) \\ (\text{cost} < -0.2) \\ (\text{cost} < -1.0) \\ (\text{cost} < -2.0) \end{array} \right\}$$

$\Longrightarrow$ SAT, $min = -2.0$



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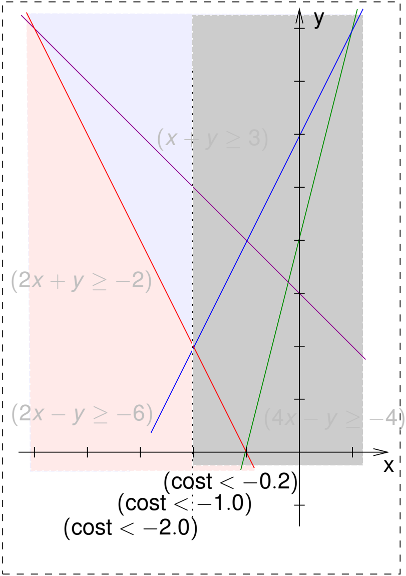
$\Longrightarrow$ UNSAT, $min = -2.0$

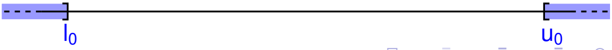


Offline Schema: Mixed Linear/Binary-Search Strategy

Input: $\langle \varphi, \text{cost}, \text{lb}, \text{ub} \rangle$ // lb can be $-\infty$, ub can be $+\infty$

$\text{l} \leftarrow \text{lb}$; $\text{u} \leftarrow \text{ub}$; $\mathcal{M} \leftarrow \emptyset$; $\varphi \leftarrow \varphi \cup \{\neg(\text{cost} < \text{lb}), (\text{cost} < \text{ub})\}$;

while $(\text{l} < \text{u})$ **do**



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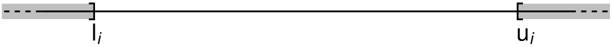
while $(\mathsf{l} < \mathsf{u})$ **do**

  if (BinSearchMode()) **then** // Binary-search Mode

  else // Linear-search Mode



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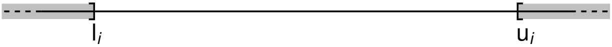
    $\langle \text{res}, \mu \rangle \leftarrow$ SMT.IncrementalSolve(φ);



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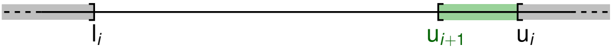
while $(\text{l} < \text{u})$ **do**
  if (BinSearchMode()) **then** // Binary-search Mode
  else // Linear-search Mode
    $\langle \text{res}, \mu \rangle \leftarrow$ SMT.IncrementalSolve(φ);
  if (res = SAT) **then**
    $\langle \mathcal{M}, \text{u} \rangle \leftarrow \mathcal{LRA}$-Solver.Minimize(cost, μ);
    $\varphi \leftarrow \varphi \cup \{(\text{cost} < \text{u})\}$;
  else {res = UNSAT}



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while $(\text{l} < \text{u})$ **do**

    if (BinSearchMode()) **then** // Binary-search Mode

    else // Linear-search Mode

        $\langle \text{res}, \mu \rangle \leftarrow$ SMT.IncrementalSolve(φ);

    if $(\text{res} = \text{SAT})$ **then**

    else $\{\text{res} = \text{UNSAT}\}$

            $\text{l} \leftarrow \text{u}$;

return $\langle \mathcal{M}, \text{u} \rangle$

Offline Schema: Mixed Linear/Binary-Search Strategy

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while $(\text{l} < \text{u})$ **do**

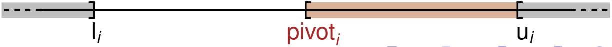
  if (BinSearchMode()) **then** // Binary-search Mode

    $\text{pivot} \leftarrow \text{ComputePivot}(\text{l}, \text{u})$;

    $\varphi \leftarrow \varphi \cup \{(\text{cost} < \text{pivot})\}$;

    $\langle\text{res}, \mu\rangle \leftarrow \text{SMT.IncrementalSolve}(\varphi)$;

  else // Linear-search Mode



Offline Schema: Mixed Linear/Binary-Search Strategy

Input: $\langle \varphi, \text{cost}, \text{lb}, \text{ub} \rangle$ // lb can be $-\infty$, ub can be $+\infty$

$\text{l} \leftarrow \text{lb}$; $\text{u} \leftarrow \text{ub}$; $\mathcal{M} \leftarrow \emptyset$; $\varphi \leftarrow \varphi \cup \{\neg(\text{cost} < \text{lb}), (\text{cost} < \text{ub})\}$;

while $(\text{l} < \text{u})$ **do**

  if (BinSearchMode()) **then** // Binary-search Mode

    $\text{pivot} \leftarrow \text{ComputePivot}(\text{l}, \text{u})$;

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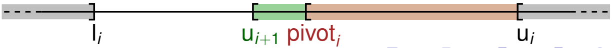
  else // Linear-search Mode

  if $(\text{res} = \text{SAT})$ **then**

    $\langle \mathcal{M}, \text{u} \rangle \leftarrow \mathcal{LRA}\text{-Solver.Minimize}(\text{cost}, \mu)$;

    $\varphi \leftarrow \varphi \cup \{(\text{cost} < \text{u})\}$;

  else $\{\text{res} = \text{UNSAT}\}$



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    if $(\text{res} = \text{SAT})$ **then**

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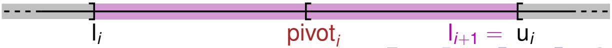
        if $((\text{cost} < \text{pivot}) \notin \text{SMT.ExtractUnsatCore}(\varphi))$ **then**

            $\text{l} \leftarrow \text{u}$;

        else

return $\langle \mathcal{M}, \text{u} \rangle$



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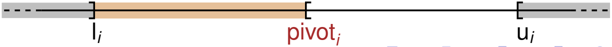
    else

      $\text{l} \leftarrow \text{pivot};$

      $\varphi \leftarrow (\varphi \setminus \{(\text{cost} < \text{pivot})\}) \cup \{\neg(\text{cost} < \text{pivot})\};$



The Minimizer

Minimizer embedded within the Simplex-based $\mathcal{LRA}$-solver by [25]

- Minimization by standard Simplex techniques

Strict Inequalities

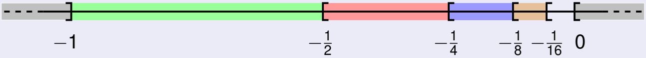
Temporally treated as non-strict inequalities:

- if minimum cost *min* lays only on non-strict inequalities, *min* is a solution
- otherwise, for some $\delta > 0$ there exists a solution for every cost $c \in]min, min + \delta]$

If *min* is a non-strict minimum, then $(cost \leq min)$ is added to φ.

Binary vs. Linear search

Beware of Zeno: pure binary search can cause infinite partitioning



- E.g. if no solution in $[-1, 0[$, then $[-1, 0[, [-1/2, 0[, [-1/4, 0[, [-1/8, 0[, \ldots$
- SMT solver may find a conflict set $\eta \cup (\text{cost} < \text{pivot})$ even if $\varphi \setminus \{(\text{cost} < \text{pivot})\}$ is $\mathcal{LRA}$-inconsistent
- Solution: Binary-search interleaved with linear-search (Mixed Linear/Binary Search Strategy)

Note: Binary search not "obviously faster" than linear search

- Binary search: typically smaller number of range-restriction steps
- Linear search: average smaller cost of each range-restriction steps (unsatisfiable calls typically much harder than sat. ones)

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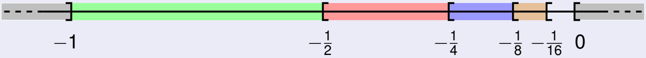

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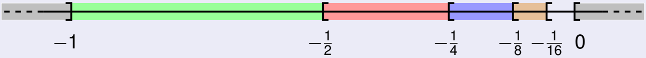

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Termination & Correctness

Termination

The linear search procedure terminates:

- Finite number of satisfiable truth assignments μ_i
- No truth assignment μ_i generated twice
  - guaranteed by computing the minimum cost m_i of μ_i and learning $(\mathrm{cost} < \mathrm{m}_i)$

$\Longrightarrow$ also the mixed linear/binary search procedure terminates

Correctness

The procedure returns the minimum cost

- Explores the whole space of satisfiable truth assignments
- For every satisfiable truth assignment, Minimize finds the minimum cost

Some Enhancements [52, 53, 16]

- After invoking the minimizer and learning $(\text{cost} < \text{m}_i)$
  - Invoke $\mathcal{LRA}$-solver.solve$(\mu_i \wedge (\text{cost} < \text{m}_i)) \Rightarrow$ *conflict set* η_i and learn also $\neg\eta_i$
  - Binary mode: learn also $(\text{cost} < \text{pivot}_i)$ to reuse previously learned clauses in the form $\neg(\text{cost} < \text{pivot}_i) \vee C$
- Tightening of conflicts on binary search [52, 53, 16])
  - when $\varphi \wedge (\text{cost} < \mathit{pivot}_i)$ fails, look for tighter conflict $\neg(\text{cost} < M_i)$ s.t. $M_i > \mathit{pivot}_i$
- Adaptive Mixed Linear/Binary-Search Strategy: BinSearchMode() chooses according to $\frac{\Delta \text{ub}}{\Delta \# \mathit{conflicts}}$

From OMT($\mathcal{LRA}$) to OMT($\mathcal{LRA} \cup \mathcal{T}$)

OMT($\mathcal{LRA}$) procedure extended for handling $\mathcal{LRA} \cup \mathcal{T}$-formulas φ:

For free if SMT solver handles $\mathcal{LRA} \cup \mathcal{T}$-solving by *Delayed Theory Combination* [18] or Model-based Combination [23], splitting negated interface equalities $\neg(x_i = x_j)$ into $((x_i < x_j) \vee (x_i > x_j))$:

- Truth assignments $\mu' \stackrel{\text{def}}{=} \mu_{\mathcal{LRA}} \cup \mu_{eid} \cup \mu_{\mathcal{T}}$ s.t. $\mu' \models \varphi$
  - μ_{eid} is a set containing interface equalities $(x_i = x_j)$, disequalities $\neg(x_i = x_j)$ and one inequality in $\{(x_i < x_j), (x_i > x_j)\}$ for every disequality in μ_{eid}
- $\mathcal{LRA}$-solver.solve invoked on $\mu'_{\mathcal{LRA}}$
  - $\mu'_{\mathcal{LRA}} \stackrel{\text{def}}{=} \mu_{\mathcal{LRA}} \cup \mu_{ei}$ obtained from μ_{eid} by dropping disequalities

$\Longrightarrow$ $\mathcal{LRA}$-solver.minimize invoked on $\langle \text{cost}, \mu'_{\mathcal{LRA}} \rangle$

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From OMT($\mathcal{LRA} \cup \mathcal{T}$) to OMT($\mathcal{LIRA} \cup \mathcal{T}$) [55, 16]

- OMT($\mathcal{LRA} \cup \mathcal{T}$) procedures extended to $\mathcal{LIA}$ and mixed $\mathcal{LRA}$/$\mathcal{LIA}$ costs [16, 55]
- $\mathcal{LRA}$/$\mathcal{LIA}$-solvers enhanced with ILP minimization techniques (branch & bound, cutting planes, backjumping, ...)
- Note: with $\mathcal{LIA}$
  - ILP minimization often expensive
  - no "Zeno" problem for binary search
  - in principle, if problem is lower-bounded, the ILP minimizer is not necessary
- tradeoff between LP, (in)complete ILP minimization, binary search and Boolean Search [16, 55]

Truncated Branch and Bound

Observations:

- branch & bound can be expensive in degenerate cases
- optimality not truly necessary

Idea:

always stop B&B after first iteration, even if cost value is not guaranteed to be optimal.

Trade-off:

- less expensive minimization procedure on Integers
- risk of CDCL generating same μ multiple times

Outline

Incremental OMT [15, 55, 54]

Call OMT incrementally

- e.g., in BMC with parametric systems [53]

Intuition

In OMT, all learned clauses are either $\mathcal{T}$-lemmas, or derive from $\mathcal{T}$-lemmas and the original formulas , or are in the form $(\text{cost} < min)$
$\Longrightarrow$ exploit incrementality of SMT solvers, in two alternative ways:

(i) drop the $(\text{cost} < min)$ from one OMT call to the other

(ii) assert fresh variable S at each OMT call, and learn $\neg S \vee (\text{cost} < min)$ instead of $(\text{cost} < min)$

$\Longrightarrow$ can reuse learned clauses from OMT call to the other, (included these in the form $\neg(\text{cost} < min_{old}) \vee C$ as soon as $min_{cur} \leq min_{old}$.)

OMT with Independent Objectives (Boxed OMT) [38, 55]

The problem: $\langle \varphi, \{\mathsf{cost}_1, ..., \mathsf{cost}_k\} \rangle$ [38]

Given $\langle \varphi, \mathcal{C} \rangle$ s.t.:

- φ is the input formula
- $\mathcal{C} \stackrel{\text{def}}{=} \{\mathsf{cost}_1, ..., \mathsf{cost}_k\}$ is a set of $\mathcal{LIRA}$-terms on variables in φ,

$\langle \varphi, \mathcal{C} \rangle$ is the problem of finding a set of independent $\mathcal{LIRA}$-models $\mathcal{M}_1, ..., \mathcal{M}_k$ s.t. s.t. each $\mathcal{M}_i$ makes cost_i minimum.

Notes

- derives from SW verification problems [38]
- equivalent to k independent problems $\langle \varphi, \mathsf{cost}_1 \rangle, ..., \langle \varphi, \mathsf{cost}_k \rangle$
- intuition: share search effort for the different objectives
- generalizes to OMT($\mathcal{LIRA} \cup \mathcal{T}$) straightforwardly

OMT with Multiple Objectives [38, 16, 55]

Solution

- Intuition: when a $\mathcal{T}$-consistent satisfying assignment μ is found,

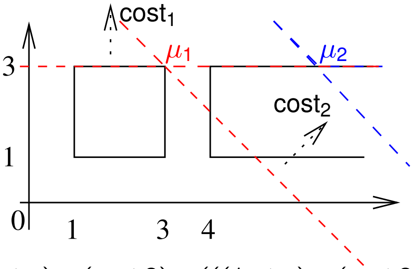
  $\texttt{foreach}\ \textbf{cost}_{\texttt{i}}$

  $\quad \texttt{min}_{\texttt{i}} := \texttt{min}\{\texttt{min}_{\texttt{i}}, \mathcal{T}\texttt{solver.minimize}(\mu, \textbf{cost}_{\texttt{i}})\};$

  $\texttt{learn}\ \bigvee_{\texttt{i}}(\textbf{cost}_{\texttt{i}} < \texttt{min}_{\texttt{i}}); \qquad // \ (\textbf{cost}_{\texttt{i}} < -\infty) \equiv \bot$

  proceed until UNSAT;
- Notice:
  - for each μ, guaranteed improvement of at least one min_i
  - in practice, for each μ, multiple cost_i minima are improved
- Implemented improvements:
  - (a) drop previous clauses $\bigvee_i(\mathsf{cost}_i < min_i)$
  - (b) $(\mathsf{cost}_i < min_i)$ pushed in μ first: if $\mathcal{T}$-inconsistent, skip minimization
  - (c) learn $\neg(\mathsf{cost}_i < min_i) \vee (\mathsf{cost}_i < min_i^{old})$, s.t. min_i^{old} previous min_i
    $\Longrightarrow$ reuse previously-learned clauses like $\neg(\mathsf{cost}_i < min_i^{old}) \vee C$

Boxed OMT: Example [38, 55]



$$\varphi \quad = \quad (1 \leq y) \wedge (y \leq 3) \wedge (((1 \leq x) \wedge (x \leq 3)) \vee (x \geq 4))$$
$$\wedge \quad (\text{cost}_1 = -y) \wedge (\text{cost}_2 = -x - y)$$

$$\mu_1 \quad = \quad \{(1 \leq y), (y \leq 3), (1 \leq x), (x \leq 3)\} \Longrightarrow \text{SAT} \Longrightarrow [-3, -6]$$

$\Longrightarrow$ *learn* $\{(\text{cost}_1 < -3) \vee (\text{cost}_2 < -6)\}$

$$\mu_2 \quad = \quad \{(1 \leq y), (y \leq 3), (x \geq 4)\} \Longrightarrow \text{SAT} \Longrightarrow [-3, -\infty]$$

$\Longrightarrow$ *learn* $\{(\text{cost}_1 < -3)\}$

$\Longrightarrow$ UNSAT

OMT with Lexicographic Combination of Objectives [16]

The problem

Find one optimal model $\mathcal{M}$ minimizing *costs* $\stackrel{\text{def}}{=}$ $\text{cost}_1, \text{cost}_2, ..., \text{cost}_k$ lexicographically.

Solution

- Intuition:

  $\{$*minimize* $\text{cost}_1\}$
  when UNSAT
  $\{$*substitute unit clause* $(\text{cost}_1 < min_1)$ *with* $(\text{cost}_1 = min_1)\}$
  $\{$*minimize* $\text{cost}_2\}$
  ...

OMT with Other forms of Objective Combination

OMT with Min-Max [Max-Min] optimization

Given $\langle \varphi, \{\text{cost}_1, ..., \text{cost}_k\}\rangle$, find a solution which minimizes the maximum value among $\{\text{cost}_1, ..., \text{cost}_k\}$. (Max-Min dual.)

- Frequent in some applications (e.g. [53, 59])

$\Longrightarrow$ encode into OMT($\mathcal{LIRA} \cup \mathcal{T}$) problem
$\{\varphi \wedge \bigwedge_i (\text{cost}_i \leq \text{cost}), \text{cost}\}$ s.t. cost fresh.

OMT with linear combinations of costs

Given $\langle \varphi, \{\text{cost}_1, ..., \text{cost}_k\}\rangle$ and a set of weights $\{w_1, ..., w_k\}$, find a solution which minimizes $\sum_i w_i \cdot \text{cost}_i$.

$\Longrightarrow$ encode into OMT($\mathcal{LIRA} \cup \mathcal{T}$) problem
$\{\varphi \wedge (\text{cost} = \sum_i w_i \cdot \text{cost}_i), \text{cost}\}$ s.t. cost fresh.

These objectives can be composed with other OMT($\mathcal{LIRA}$) objectives.

Outline

OMT($\mathcal{LRA} \cup \mathcal{T}$) vs. SMT with PB costs (& MaxSMT)

SMT + PB costs (& MaxSMT) can be encoded into OMT($\mathcal{LRA} \cup \mathcal{T}$):

$$\begin{array}{ll} \text{minimize} & \sum_j w_j \cdot A_j \quad //(\sum_j ite(A_j, w_j, 0)) \\ \text{s.t.} & \varphi \\ \Downarrow & \\ \text{minimize} & \sum_j x_j \\ \text{s.t.} & \varphi \wedge \bigwedge_j (A_j \rightarrow (x_j = w_j)) \wedge (\neg A_j \rightarrow (x_j = 0)) \\ & \quad \wedge \bigwedge_j ((x_j \geq 0) \wedge (x_j \leq w_j)) \end{array}$$

but not vice versa!

- SMT + PB costs finds the minimum-cost $\mathcal{T}$-satisfiable assignment
  $\Longrightarrow$ search for minimum is purely Boolean
- OMT($\mathcal{LIRA} \cup \mathcal{T}$) finds the $\mathcal{T}$-satisfiable assignment whose minimum cost is minimum
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Remark: range constraints "$(x_j \geq 0) \wedge (x_j \leq w_j)$"

$$\begin{array}{ll} OMT+PB: & \sum_j w_j \cdot A_j,\ w_i > 0\ \ //(\sum_j ite(A_j, w_j, 0)) \\ & \Downarrow \\ & \sum_j x_j,\ x_j\ fresh \\ \text{s.t.} & \ldots \wedge \bigwedge_j (A_j \to (x_j = w_j)) \wedge (\neg A_j \to (x_j = 0)) \\ & \wedge (x_j \geq 0) \wedge (x_j \leq w_j) \end{array}$$

Range constraints "$(x_j \geq 0) \wedge (x_j \leq w_j)$" logically redundant, but essential for efficiency:

- Without range constraints, the SMT solver can detect the violation of a bound only after all A_i's are assigned :
  Ex: $w_1 = 4$, $w_2 = 7$, $\sum_{i=1} x_i < 10$, $A_1 = A_2 = \top$, $A_i = * \ \forall i > 2$.
- With range constraints, the SMT solver detects the violation as soon as the assigned A_i's violate a bound
  $\Longrightarrow$ drastic pruning of the search

Further improvement: Enhance encoding of PB constraints/MaxSMT with sorting networks [56]

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SMT/OMT with Pseudo-Boolean Costraints & Costs:

Alternative Solution: conversion into SMT($\mathcal{T}$)

- SAT + PB can be efficiently encoded into SAT [26]

$\Longrightarrow$ encode SMT($\mathcal{T}$) + PB into SMT($\mathcal{T}$)

- similar idea implemented in [16, 15] for cardinality constraints

Alternative Solution: Leverage SAT+PB

- develop a “modulo theory” version of your favourite PB-solver
- afaik, no implementation available

Alternative Solution: SMT($\mathcal{T} \cup \mathcal{C}$) [20]

- $\mathcal{C}$ is an ad-hoc “theory of costs”
- a specialized very-fast theory-solver for $\mathcal{C}$ added
  - very fast & aggressive search pruning and theory-propagation

SMT/OMT with Pseudo-Boolean Costraints & Costs:

Alternative Solution: conversion into SMT($\mathcal{T}$)

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$\Longrightarrow$ encode SMT($\mathcal{T}$) + PB into SMT($\mathcal{T}$)

- similar idea implemented in [16, 15] for cardinality constraints

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- develop a “modulo theory” version of your favourite PB-solver
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Alternative Solution: SMT($\mathcal{T} \cup \mathcal{C}$) [20]

- $\mathcal{C}$ is an ad-hoc “theory of costs”
- a specialized very-fast theory-solver for $\mathcal{C}$ added
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A "Theory of cost" $\mathcal{C}$

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- M variables $cost^i$
- predicate "bound cost" $BC(cost^i, k)$ ("$cost^i \leq k$")
- predicate "incur cost" $IC(cost^i, j, \mathrm{c}^i_j)$ ("the jth addend of $cost^i$ is c^i_j")
- "$cost^i = \sum_{j=1}^{N^i} \mathrm{c}^i_j \cdot A^i_j, \quad s.t.\ cost^i \ \in \ (l^i, u^i]$"
  encoded as:
  $\neg BC(cost^i, l^i) \wedge BC(cost^i, u^i) \wedge \bigwedge_{j=1}^{N^i} (A^i_j \leftrightarrow IC(cost^i, j, \mathrm{c}^i_j))$

$\mathcal{C}$-solver

for each i, $\mathcal{C}$-solver mantains the current values of the incurred costs $cost^i \stackrel{\text{def}}{=} \sum_{IC(cost^i,j,c^i_j) \leftarrow \top} c^i_j$, the total cost of all unassigned IC's $\Delta cost^i \stackrel{\text{def}}{=} \sum_{\{IC(cost^i,j,c^i_j)\ unassigned\}} c^i_j$, and of the range $]lb_i, ub^i]$

1. $BC(cost^i, c) \leftarrow \top/\bot \Longrightarrow$ update $]lb_i, ub^i]$
2. $IC(cost^i, j, c^i_j) \leftarrow \top \Longrightarrow cost^i \leftarrow cost^i + c^i_j$
   $IC(cost^i, j, c^i_j) \leftarrow \bot \Longrightarrow \Delta cost^i \leftarrow \Delta cost^i - c^i_j$
3. $cost^i > ub^i \Longrightarrow$ conflict
4. $cost^i + \Delta cost^i \leq lb^i \Longrightarrow$ conflict
5. $IC(cost^i, j, c^i_j) \leftarrow \top$ causes 3. $\Longrightarrow$ propagate $\neg IC(cost^i, j, c^i_j)$
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- very fast:
  - add one constraint & solve: 1 sum + 1 comparison
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MaxSAT Modulo Theories (MaxSMT) I

[Partial Weighted] MaxSMT: The problem

Input: $\varphi_h^{\mathcal{T}}, \varphi_s^{\mathcal{T}}$: resp. sets of hard and (weighted) soft $\mathcal{T}$-clauses;

Output: a maximum-weight set of soft $\mathcal{T}$-clauses $\psi_s^{\mathcal{T}}$ s.t. $\psi_s^{\mathcal{T}} \subseteq \varphi_s^{\mathcal{T}}$ and $\varphi_h^{\mathcal{T}} \cup \psi_s^{\mathcal{T}}$ is $\mathcal{T}$-satisfiable

MaxSMT vs. SMT with PB cost functions

MaxSMT $\langle \varphi_h^{\mathcal{T}}, \varphi_s^{\mathcal{T}} \rangle$ encodable into SMT with PB costs $\langle \varphi^{\mathcal{T}\prime}, \text{cost} \rangle$:

$$\varphi^{\mathcal{T}\prime} \stackrel{\text{def}}{=} \varphi_h^{\mathcal{T}} \cup \bigcup_{C_j^{\mathcal{T}} \in \varphi_s^{\mathcal{T}}} \{(A_j \vee C_j^{\mathcal{T}})\}; \quad \text{cost} \stackrel{\text{def}}{=} \sum_{C_j^{\mathcal{T}} \in \varphi_s^{\mathcal{T}}} w_j \cdot A_j,$$

SMT with PB costs $\langle \varphi^{\mathcal{T}\prime}, \text{cost} \stackrel{\text{def}}{=} \sum_j w_j \cdot A_j \rangle$ encodable into MaxSMT:

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Solution: encode into OMT($\mathcal{LRA}$) [44, 52, 53]

- can be composed with other objective functions

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- Idea: Combine an SMT and a MaxSAT solver:
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Input: $\varphi_h^{\mathcal{T}}, \varphi_s^{\mathcal{T}}$ // sets of hard and (weighted) soft $\mathcal{T}$-clauses
$\langle \varphi_h^{\mathcal{B}}, \varphi_s^{\mathcal{B}} \rangle \leftarrow \mathcal{T}2\mathcal{B}\ (\langle \varphi_h^{\mathcal{T}}, \varphi_s^{\mathcal{T}} \rangle)$;
$\Theta^{\mathcal{T}} \leftarrow \emptyset$; // current set of $\mathcal{T}$-lemmas
$\psi_s^{\mathcal{T}} \leftarrow \varphi_s^{\mathcal{T}}$; // current approximation of the result
while (SMT.Solve$(\varphi_h^{\mathcal{T}} \cup \psi_s^{\mathcal{T}} \cup \Theta^{\mathcal{T}})$ = UNSAT) **do**
  $\Theta^{\mathcal{T}} \leftarrow \Theta^{\mathcal{T}} \cup$ SMT.GetTLemmas(); $\Theta^{\mathcal{B}} \leftarrow \mathcal{T}2\mathcal{B}\ (\Theta^{\mathcal{T}})$;
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return $\psi_s^{\mathcal{T}}$;

Based on the cyclic interaction of an SMT and a MaxSAT solver:

- SMT.Solve used as a generator of sets of $\mathcal{T}$-lemmas $\Theta_0^{\mathcal{T}}, \Theta_1^{\mathcal{T}}, \dots$
  $\Longrightarrow$ provide the information to rule-out $\mathcal{T}$-inconsistent solutions
- MaxSAT used to extract minimum-cost clause sets $\psi_{s,0}^{\mathcal{B}}, \psi_{s,1}^{\mathcal{B}}, \dots$
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A toy example I

$$\varphi_h^{\mathcal{T}} \stackrel{\text{def}}{=} \emptyset \qquad \varphi_h^{\mathcal{B}} \stackrel{\text{def}}{=} \emptyset$$

$$\varphi_s^{\mathcal{T}} \stackrel{\text{def}}{=} \left\{ \begin{array}{lll} C_0 : & ((x \leq 0)) & [4] \\ C_1 : & ((x \leq 1)) & [3] \\ C_2 : & ((x \geq 2)) & [2] \\ C_3 : & ((x \geq 3)) & [6] \end{array} \right\} \qquad \varphi_s^{\mathcal{B}} \stackrel{\text{def}}{=} \left\{ \begin{array}{ll} (A_0) & [4] \\ (A_1) & [3] \\ (A_2) & [2] \\ (A_3) & [6] \end{array} \right\}$$

Notice that the set of all (minimal) $\mathcal{T}$-lemmas on the $\mathcal{T}$-atoms of $\varphi_h^{\mathcal{T}} \cup \varphi_s^{\mathcal{T}}$ is:

$$\Theta_*^{\mathcal{T}} = \left\{ \begin{array}{ll} \theta_1 : & (\neg(x \leq 0) \vee (x \leq 1)) \\ \theta_2 : & (\neg(x \geq 3) \vee (x \geq 2)) \\ \theta_3 : & (\neg(x \leq 0) \vee \neg(x \geq 2)) \\ \theta_4 : & (\neg(x \leq 0) \vee \neg(x \geq 3)) \\ \theta_5 : & (\neg(x \leq 1) \vee \neg(x \geq 2)) \\ \theta_6 : & (\neg(x \leq 1) \vee \neg(x \geq 3)) \end{array} \right\} \qquad \Theta_*^{\mathcal{B}} = \left\{ \begin{array}{l} (\neg A_0 \vee A_1) \\ (\neg A_3 \vee A_2) \\ (\neg A_0 \vee \neg A_2) \\ (\neg A_0 \vee \neg A_3) \\ (\neg A_1 \vee \neg A_2) \\ (\neg A_1 \vee \neg A_3) \end{array} \right\}$$

An "unlucky" possible execution of the algorithm is:

i	$\Theta_i^{\mathcal{T}}$	$\psi_{s,i}^{\mathcal{T}}$	Weight$(\psi_{s,i}^{\mathcal{T}})$	$SMT(\varphi_h^{\mathcal{T}} \cup \psi_{s,i}^{\mathcal{T}} \cup \Theta_i^{\mathcal{T}})$
0	$\{\}$	$\{C_0, C_1, C_2, C_3\}$	15	UNSAT
1	$\{\theta_4\}$	$\{\ \ , C_1, C_2, C_3\}$	11	UNSAT
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$$\Theta_*^{\mathcal{T}} = \left\{ \begin{array}{ll} \theta_1: & (\neg(x \le 0) \vee (x \le 1)) \\ \theta_2: & (\neg(x \ge 3) \vee (x \ge 2)) \\ \theta_3: & (\neg(x \le 0) \vee \neg(x \ge 2)) \\ \theta_4: & (\neg(x \le 0) \vee \neg(x \ge 3)) \\ \theta_5: & (\neg(x \le 1) \vee \neg(x \ge 2)) \\ \theta_6: & (\neg(x \le 1) \vee \neg(x \ge 3)) \end{array} \right\} \qquad \Theta_*^{\mathcal{B}} = \left\{ \begin{array}{l} (\neg A_0 \vee A_1) \\ (\neg A_3 \vee A_2) \\ (\neg A_0 \vee \neg A_2) \\ (\neg A_0 \vee \neg A_3) \\ (\neg A_1 \vee \neg A_2) \\ (\neg A_1 \vee \neg A_3) \end{array} \right\}$$

An "unlucky" possible execution of the algorithm is:

i	$\Theta_i^{\mathcal{T}}$	$\psi_{s,i}^{\mathcal{T}}$	Weight($\psi_{s,i}^{\mathcal{T}}$)	$SMT(\varphi_h^{\mathcal{T}} \cup \psi_{s,i}^{\mathcal{T}} \cup \Theta_i^{\mathcal{T}})$
0	$\{\}$	$\{C_0, C_1, C_2, C_3\}$	15	UNSAT
1	$\{\theta_4\}$	$\{\ \ , C_1, C_2, C_3\}$	11	UNSAT
2	$\{\theta_4, \theta_6\}$	$\{C_0, C_1, C_2, \ \ \}$	9	UNSAT
3	$\{\theta_4, \theta_6, \theta_3\}$	$\{\ \ , \ \ , C_2, C_3\}$	8	SAT

A toy example I

$$\varphi_h^{\mathcal{T}} \stackrel{\text{def}}{=} \emptyset \qquad \varphi_h^{\mathcal{B}} \stackrel{\text{def}}{=} \emptyset$$

$$\varphi_s^{\mathcal{T}} \stackrel{\text{def}}{=} \left\{ \begin{array}{lll} C_0 : & ((x \leq 0)) & [4] \\ C_1 : & ((x \leq 1)) & [3] \\ C_2 : & ((x \geq 2)) & [2] \\ C_3 : & ((x \geq 3)) & [6] \end{array} \right\} \qquad \varphi_s^{\mathcal{B}} \stackrel{\text{def}}{=} \left\{ \begin{array}{ll} (A_0) & [4] \\ (A_1) & [3] \\ (A_2) & [2] \\ (A_3) & [6] \end{array} \right\}$$

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An "unlucky" possible execution of the algorithm is:

i	$\Theta_i^{\mathcal{T}}$	$\psi_{s,i}^{\mathcal{T}}$	Weight$(\psi_{s,i}^{\mathcal{T}})$	$SMT(\varphi_h^{\mathcal{T}} \cup \psi_{s,i}^{\mathcal{T}} \cup \Theta_i^{\mathcal{T}})$
0	$\{\}$	$\{C_0, C_1, C_2, C_3\}$	15	UNSAT
1	$\{\theta_4\}$	$\{\ \ , C_1, C_2, C_3\}$	11	UNSAT
2	$\{\theta_4, \theta_6\}$	$\{C_0, C_1, C_2, \ \ \}$	9	UNSAT
3	$\{\theta_4, \theta_6, \theta_3\}$	$\{\ \ ,\ \ , C_2, C_3\}$	8	SAT

A toy example I

$$\varphi_h^{\mathcal{T}} \stackrel{\text{def}}{=} \emptyset \qquad \varphi_h^{\mathcal{B}} \stackrel{\text{def}}{=} \emptyset$$

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An "unlucky" possible execution of the algorithm is:

i	$\Theta_i^{\mathcal{T}}$	$\psi_{s,i}^{\mathcal{T}}$	Weight($\psi_{s,i}^{\mathcal{T}}$)	$SMT(\varphi_h^{\mathcal{T}} \cup \psi_{s,i}^{\mathcal{T}} \cup \Theta_i^{\mathcal{T}})$
0	$\{\}$	$\{C_0, C_1, C_2, C_3\}$	15	UNSAT
1	$\{\theta_4\}$	$\{\ \ , C_1, C_2, C_3\}$	11	UNSAT
2	$\{\theta_4, \theta_6\}$	$\{C_0, C_1, C_2,\ \ \}$	9	UNSAT
3	$\{\theta_4, \theta_6, \theta_3\}$	$\{\ \ ,\ \ , C_2, C_3\}$	8	SAT

A toy example I

$$\varphi_h^{\mathcal{T}} \stackrel{\text{def}}{=} \emptyset \qquad \varphi_h^{\mathcal{B}} \stackrel{\text{def}}{=} \emptyset$$

$$\varphi_s^{\mathcal{T}} \stackrel{\text{def}}{=} \left\{ \begin{array}{lll} C_0: & ((x \leq 0)) & [4] \\ C_1: & ((x \leq 1)) & [3] \\ C_2: & ((x \geq 2)) & [2] \\ C_3: & ((x \geq 3)) & [6] \end{array} \right\} \qquad \varphi_s^{\mathcal{B}} \stackrel{\text{def}}{=} \left\{ \begin{array}{ll} (A_0) & [4] \\ (A_1) & [3] \\ (A_2) & [2] \\ (A_3) & [6] \end{array} \right\}$$

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$$\Theta_*^{\mathcal{T}} = \left\{ \begin{array}{ll} \theta_1: & (\neg(x \leq 0) \vee (x \leq 1)) \\ \theta_2: & (\neg(x \geq 3) \vee (x \geq 2)) \\ \theta_3: & (\neg(x \leq 0) \vee \neg(x \geq 2)) \\ \theta_4: & (\neg(x \leq 0) \vee \neg(x \geq 3)) \\ \theta_5: & (\neg(x \leq 1) \vee \neg(x \geq 2)) \\ \theta_6: & (\neg(x \leq 1) \vee \neg(x \geq 3)) \end{array} \right\} \qquad \Theta_*^{\mathcal{B}} = \left\{ \begin{array}{l} (\neg A_0 \vee A_1) \\ (\neg A_3 \vee A_2) \\ (\neg A_0 \vee \neg A_2) \\ (\neg A_0 \vee \neg A_3) \\ (\neg A_1 \vee \neg A_2) \\ (\neg A_1 \vee \neg A_3) \end{array} \right\}$$

An "unlucky" possible execution of the algorithm is:

i	$\Theta_i^{\mathcal{T}}$	$\psi_{s,i}^{\mathcal{T}}$	Weight$(\psi_{s,i}^{\mathcal{T}})$	$SMT(\varphi_h^{\mathcal{T}} \cup \psi_{s,i}^{\mathcal{T}} \cup \Theta_i^{\mathcal{T}})$
0	$\{\}$	$\{C_0, C_1, C_2, C_3\}$	15	UNSAT
1	$\{\theta_4\}$	$\{\ \ , C_1, C_2, C_3\}$	11	UNSAT
2	$\{\theta_4, \theta_6\}$	$\{C_0, C_1, C_2,\ \ \}$	9	UNSAT
3	$\{\theta_4, \theta_6, \theta_3\}$	$\{\ \ ,\ \ , C_2, C_3\}$	8	SAT

A toy example I

$$\varphi_h^{\mathcal{T}} \stackrel{\text{def}}{=} \emptyset \qquad \varphi_h^{\mathcal{B}} \stackrel{\text{def}}{=} \emptyset$$

$$\varphi_s^{\mathcal{T}} \stackrel{\text{def}}{=} \left\{ \begin{array}{lll} C_0: & ((x \leq 0)) & [4] \\ C_1: & ((x \leq 1)) & [3] \\ C_2: & ((x \geq 2)) & [2] \\ C_3: & ((x \geq 3)) & [6] \end{array} \right\} \qquad \varphi_s^{\mathcal{B}} \stackrel{\text{def}}{=} \left\{ \begin{array}{ll} (A_0) & [4] \\ (A_1) & [3] \\ (A_2) & [2] \\ (A_3) & [6] \end{array} \right\}$$

Notice that the set of all (minimal) $\mathcal{T}$-lemmas on the $\mathcal{T}$-atoms of $\varphi_h^{\mathcal{T}} \cup \varphi_s^{\mathcal{T}}$ is:

$$\Theta_*^{\mathcal{T}} = \left\{ \begin{array}{ll} \theta_1: & (\neg(x \leq 0) \vee (x \leq 1)) \\ \theta_2: & (\neg(x \geq 3) \vee (x \geq 2)) \\ \theta_3: & (\neg(x \leq 0) \vee \neg(x \geq 2)) \\ \theta_4: & (\neg(x \leq 0) \vee \neg(x \geq 3)) \\ \theta_5: & (\neg(x \leq 1) \vee \neg(x \geq 2)) \\ \theta_6: & (\neg(x \leq 1) \vee \neg(x \geq 3)) \end{array} \right\} \qquad \Theta_*^{\mathcal{B}} = \left\{ \begin{array}{l} (\neg A_0 \vee A_1) \\ (\neg A_3 \vee A_2) \\ (\neg A_0 \vee \neg A_2) \\ (\neg A_0 \vee \neg A_3) \\ (\neg A_1 \vee \neg A_2) \\ (\neg A_1 \vee \neg A_3) \end{array} \right\}$$

An "unlucky" possible execution of the algorithm is:

i	$\Theta_i^{\mathcal{T}}$	$\psi_{s,i}^{\mathcal{T}}$	Weight$(\psi_{s,i}^{\mathcal{T}})$	$SMT(\varphi_h^{\mathcal{T}} \cup \psi_{s,i}^{\mathcal{T}} \cup \Theta_i^{\mathcal{T}})$
0	$\{\}$	$\{C_0, C_1, C_2, C_3\}$	15	UNSAT
1	$\{\theta_4\}$	$\{\ \ , C_1, C_2, C_3\}$	11	UNSAT
2	$\{\theta_4, \theta_6\}$	$\{C_0, C_1, C_2, \ \ \}$	9	UNSAT
3	$\{\theta_4, \theta_6, \theta_3\}$	$\{\ \ , \ \ , C_2, C_3\}$	8	SAT

A toy example I

$$\varphi_h^{\mathcal{T}} \stackrel{\text{def}}{=} \emptyset \qquad\qquad \varphi_h^{\mathcal{B}} \stackrel{\text{def}}{=} \emptyset$$

$$\varphi_s^{\mathcal{T}} \stackrel{\text{def}}{=} \left\{ \begin{array}{lll} C_0: & ((x \le 0)) & [4] \\ C_1: & ((x \le 1)) & [3] \\ C_2: & ((x \ge 2)) & [2] \\ C_3: & ((x \ge 3)) & [6] \end{array} \right\} \qquad \varphi_s^{\mathcal{B}} \stackrel{\text{def}}{=} \left\{ \begin{array}{ll} (A_0) & [4] \\ (A_1) & [3] \\ (A_2) & [2] \\ (A_3) & [6] \end{array} \right\}$$

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An "unlucky" possible execution of the algorithm is:

i	$\Theta_i^{\mathcal{T}}$	$\psi_{s,i}^{\mathcal{T}}$	Weight($\psi_{s,i}^{\mathcal{T}}$)	$SMT(\varphi_h^{\mathcal{T}} \cup \psi_{s,i}^{\mathcal{T}} \cup \Theta_i^{\mathcal{T}})$
0	$\{\}$	$\{C_0, C_1, C_2, C_3\}$	15	UNSAT
1	$\{\theta_4\}$	$\{\quad, C_1, C_2, C_3\}$	11	UNSAT
2	$\{\theta_4, \theta_6\}$	$\{C_0, C_1, C_2, \quad\}$	9	UNSAT
3	$\{\theta_4, \theta_6, \theta_3\}$	$\{\quad, \quad, C_2, C_3\}$	8	SAT

A toy example II

$$\varphi_h^{\mathcal{T}} \stackrel{\text{def}}{=} \emptyset \qquad \varphi_h^{\mathcal{B}} \stackrel{\text{def}}{=} \emptyset$$

$$\varphi_s^{\mathcal{T}} \stackrel{\text{def}}{=} \left\{ \begin{array}{lll} C_0 : & ((x \leq 0)) & [4] \\ C_1 : & ((x \leq 1)) & [3] \\ C_2 : & ((x \geq 2)) & [2] \\ C_3 : & ((x \geq 3)) & [6] \end{array} \right\} \qquad \varphi_s^{\mathcal{B}} \stackrel{\text{def}}{=} \left\{ \begin{array}{ll} (A_0) & [4] \\ (A_1) & [3] \\ (A_2) & [2] \\ (A_3) & [6] \end{array} \right\}$$

Notice that the set of all (minimal) $\mathcal{T}$-lemmas on the $\mathcal{T}$-atoms of $\varphi_h^{\mathcal{T}} \cup \varphi_s^{\mathcal{T}}$ is:

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A "lucky" possible execution of the algorithm is:

i	$\Theta_i^{\mathcal{T}}$	$\psi_{s,i}^{\mathcal{T}}$	Weight$(\psi_{s,i}^{\mathcal{T}})$	$SMT(\varphi_h^{\mathcal{T}} \cup \psi_{s,i}^{\mathcal{T}} \cup \Theta_i^{\mathcal{T}})$
0	$\{\}$	$\{C_0, C_1, C_2, C_3\}$	15	UNSAT
1	$\{\theta_1, \theta_2, \theta_5\}$	$\{\quad,\quad, C_2, C_3\}$	8	SAT

A toy example II

$$\varphi_h^{\mathcal{T}} \stackrel{\text{def}}{=} \emptyset \qquad \varphi_h^{\mathcal{B}} \stackrel{\text{def}}{=} \emptyset$$

$$\varphi_s^{\mathcal{T}} \stackrel{\text{def}}{=} \left\{ \begin{array}{lll} C_0 : & ((x \leq 0)) & [4] \\ C_1 : & ((x \leq 1)) & [3] \\ C_2 : & ((x \geq 2)) & [2] \\ C_3 : & ((x \geq 3)) & [6] \end{array} \right\} \qquad \varphi_s^{\mathcal{B}} \stackrel{\text{def}}{=} \left\{ \begin{array}{ll} (A_0) & [4] \\ (A_1) & [3] \\ (A_2) & [2] \\ (A_3) & [6] \end{array} \right\}$$

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$$\Theta_*^{\mathcal{T}} = \left\{ \begin{array}{ll} \theta_1 : & (\neg(x \leq 0) \vee (x \leq 1)) \\ \theta_2 : & (\neg(x \geq 3) \vee (x \geq 2)) \\ \theta_3 : & (\neg(x \leq 0) \vee \neg(x \geq 2)) \\ \theta_4 : & (\neg(x \leq 0) \vee \neg(x \geq 3)) \\ \theta_5 : & (\neg(x \leq 1) \vee \neg(x \geq 2)) \\ \theta_6 : & (\neg(x \leq 1) \vee \neg(x \geq 3)) \end{array} \right\} \qquad \Theta_*^{\mathcal{B}} = \left\{ \begin{array}{l} (\neg A_0 \vee A_1) \\ (\neg A_3 \vee A_2) \\ (\neg A_0 \vee \neg A_2) \\ (\neg A_0 \vee \neg A_3) \\ (\neg A_1 \vee \neg A_2) \\ (\neg A_1 \vee \neg A_3) \end{array} \right\}$$

A "lucky" possible execution of the algorithm is:

i	$\Theta_i^{\mathcal{T}}$	$\psi_{s,i}^{\mathcal{T}}$	Weight($\psi_{s,i}^{\mathcal{T}}$)	$SMT(\varphi_h^{\mathcal{T}} \cup \psi_{s,i}^{\mathcal{T}} \cup \Theta_i^{\mathcal{T}})$
0	$\{\}$	$\{C_0, C_1, C_2, C_3\}$	15	UNSAT
1	$\{\theta_1, \theta_2, \theta_5\}$	$\{\quad , \quad , C_2, C_3\}$	8	SAT

A toy example II

$$\varphi_h^{\mathcal{T}} \stackrel{\text{def}}{=} \emptyset \qquad\qquad \varphi_h^{\mathcal{B}} \stackrel{\text{def}}{=} \emptyset$$

$$\varphi_s^{\mathcal{T}} \stackrel{\text{def}}{=} \left\{ \begin{array}{lll} C_0 : & ((x \leq 0)) & [4] \\ C_1 : & ((x \leq 1)) & [3] \\ C_2 : & ((x \geq 2)) & [2] \\ C_3 : & ((x \geq 3)) & [6] \end{array} \right\} \qquad \varphi_s^{\mathcal{B}} \stackrel{\text{def}}{=} \left\{ \begin{array}{ll} (A_0) & [4] \\ (A_1) & [3] \\ (A_2) & [2] \\ (A_3) & [6] \end{array} \right\}$$

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i	$\Theta_i^{\mathcal{T}}$	$\psi_{s,i}^{\mathcal{T}}$	Weight($\psi_{s,i}^{\mathcal{T}}$)	$SMT(\varphi_h^{\mathcal{T}} \cup \psi_{s,i}^{\mathcal{T}} \cup \Theta_i^{\mathcal{T}})$
0	$\{\}$	$\{C_0, C_1, C_2, C_3\}$	15	UNSAT
1	$\{\theta_1, \theta_2, \theta_5\}$	$\{\quad, \quad, C_2, C_3\}$	8	SAT

A toy example II

$$\varphi_h^{\mathcal{T}} \stackrel{\text{def}}{=} \emptyset \qquad \varphi_h^{\mathcal{B}} \stackrel{\text{def}}{=} \emptyset$$

$$\varphi_s^{\mathcal{T}} \stackrel{\text{def}}{=} \left\{ \begin{array}{lll} C_0 : & ((x \leq 0)) & [4] \\ C_1 : & ((x \leq 1)) & [3] \\ C_2 : & ((x \geq 2)) & [2] \\ C_3 : & ((x \geq 3)) & [6] \end{array} \right\} \qquad \varphi_s^{\mathcal{B}} \stackrel{\text{def}}{=} \left\{ \begin{array}{ll} (A_0) & [4] \\ (A_1) & [3] \\ (A_2) & [2] \\ (A_3) & [6] \end{array} \right\}$$

Notice that the set of all (minimal) $\mathcal{T}$-lemmas on the $\mathcal{T}$-atoms of $\varphi_h^{\mathcal{T}} \cup \varphi_s^{\mathcal{T}}$ is:

$$\Theta_*^{\mathcal{T}} = \left\{ \begin{array}{ll} \theta_1 : & (\neg(x \leq 0) \vee (x \leq 1)) \\ \theta_2 : & (\neg(x \geq 3) \vee (x \geq 2)) \\ \theta_3 : & (\neg(x \leq 0) \vee \neg(x \geq 2)) \\ \theta_4 : & (\neg(x \leq 0) \vee \neg(x \geq 3)) \\ \theta_5 : & (\neg(x \leq 1) \vee \neg(x \geq 2)) \\ \theta_6 : & (\neg(x \leq 1) \vee \neg(x \geq 3)) \end{array} \right\} \qquad \Theta_*^{\mathcal{B}} = \left\{ \begin{array}{l} (\neg A_0 \vee A_1) \\ (\neg A_3 \vee A_2) \\ (\neg A_0 \vee \neg A_2) \\ (\neg A_0 \vee \neg A_3) \\ (\neg A_1 \vee \neg A_2) \\ (\neg A_1 \vee \neg A_3) \end{array} \right\}$$

A "lucky" possible execution of the algorithm is:

i	$\Theta_i^{\mathcal{T}}$	$\psi_{s,i}^{\mathcal{T}}$	Weight$(\psi_{s,i}^{\mathcal{T}})$	$SMT(\varphi_h^{\mathcal{T}} \cup \psi_{s,i}^{\mathcal{T}} \cup \Theta_i^{\mathcal{T}})$
0	$\{\}$	$\{C_0, C_1, C_2, C_3\}$	15	UNSAT
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A toy example II

$$\varphi_h^{\mathcal{T}} \stackrel{\text{def}}{=} \emptyset \qquad \varphi_h^{\mathcal{B}} \stackrel{\text{def}}{=} \emptyset$$

$$\varphi_s^{\mathcal{T}} \stackrel{\text{def}}{=} \left\{ \begin{array}{lll} C_0: & ((x \leq 0)) & [4] \\ C_1: & ((x \leq 1)) & [3] \\ C_2: & ((x \geq 2)) & [2] \\ C_3: & ((x \geq 3)) & [6] \end{array} \right\} \qquad \varphi_s^{\mathcal{B}} \stackrel{\text{def}}{=} \left\{ \begin{array}{ll} (A_0) & [4] \\ (A_1) & [3] \\ (A_2) & [2] \\ (A_3) & [6] \end{array} \right\}$$

Notice that the set of all (minimal) $\mathcal{T}$-lemmas on the $\mathcal{T}$-atoms of $\varphi_h^{\mathcal{T}} \cup \varphi_s^{\mathcal{T}}$ is:

$$\Theta_*^{\mathcal{T}} = \left\{ \begin{array}{ll} \theta_1: & (\neg(x \leq 0) \vee (x \leq 1)) \\ \theta_2: & (\neg(x \geq 3) \vee (x \geq 2)) \\ \theta_3: & (\neg(x \leq 0) \vee \neg(x \geq 2)) \\ \theta_4: & (\neg(x \leq 0) \vee \neg(x \geq 3)) \\ \theta_5: & (\neg(x \leq 1) \vee \neg(x \geq 2)) \\ \theta_6: & (\neg(x \leq 1) \vee \neg(x \geq 3)) \end{array} \right\} \qquad \Theta_*^{\mathcal{B}} = \left\{ \begin{array}{l} (\neg A_0 \vee A_1) \\ (\neg A_3 \vee A_2) \\ (\neg A_0 \vee \neg A_2) \\ (\neg A_0 \vee \neg A_3) \\ (\neg A_1 \vee \neg A_2) \\ (\neg A_1 \vee \neg A_3) \end{array} \right\}$$

A "lucky" possible execution of the algorithm is:

i	$\Theta_i^{\mathcal{T}}$	$\psi_{s,i}^{\mathcal{T}}$	Weight($\psi_{s,i}^{\mathcal{T}}$)	$SMT(\varphi_h^{\mathcal{T}} \cup \psi_{s,i}^{\mathcal{T}} \cup \Theta_i^{\mathcal{T}})$
0	$\{\}$	$\{C_0, C_1, C_2, C_3\}$	15	UNSAT
1	$\{\theta_1, \theta_2, \theta_5\}$	$\{\ \ ,\ \ , C_2, C_3\}$	8	SAT

A toy example II

$$\varphi_h^{\mathcal{T}} \stackrel{\text{def}}{=} \emptyset \qquad\qquad \varphi_h^{\mathcal{B}} \stackrel{\text{def}}{=} \emptyset$$

$$\varphi_s^{\mathcal{T}} \stackrel{\text{def}}{=} \left\{ \begin{array}{lll} C_0: & ((x \leq 0)) & [4] \\ C_1: & ((x \leq 1)) & [3] \\ C_2: & ((x \geq 2)) & [2] \\ C_3: & ((x \geq 3)) & [6] \end{array} \right\} \qquad \varphi_s^{\mathcal{B}} \stackrel{\text{def}}{=} \left\{ \begin{array}{ll} (A_0) & [4] \\ (A_1) & [3] \\ (A_2) & [2] \\ (A_3) & [6] \end{array} \right\}$$

Notice that the set of all (minimal) $\mathcal{T}$-lemmas on the $\mathcal{T}$-atoms of $\varphi_h^{\mathcal{T}} \cup \varphi_s^{\mathcal{T}}$ is:

$$\Theta_*^{\mathcal{T}} = \left\{ \begin{array}{ll} \theta_1: & (\neg(x \leq 0) \vee (x \leq 1)) \\ \theta_2: & (\neg(x \geq 3) \vee (x \geq 2)) \\ \theta_3: & (\neg(x \leq 0) \vee \neg(x \geq 2)) \\ \theta_4: & (\neg(x \leq 0) \vee \neg(x \geq 3)) \\ \theta_5: & (\neg(x \leq 1) \vee \neg(x \geq 2)) \\ \theta_6: & (\neg(x \leq 1) \vee \neg(x \geq 3)) \end{array} \right\} \qquad \Theta_*^{\mathcal{B}} = \left\{ \begin{array}{l} (\neg A_0 \vee A_1) \\ (\neg A_3 \vee A_2) \\ (\neg A_0 \vee \neg A_2) \\ (\neg A_0 \vee \neg A_3) \\ (\neg A_1 \vee \neg A_2) \\ (\neg A_1 \vee \neg A_3) \end{array} \right\}$$

A "lucky" possible execution of the algorithm is:

i	$\Theta_i^{\mathcal{T}}$	$\psi_{s,i}^{\mathcal{T}}$	Weight$(\psi_{s,i}^{\mathcal{T}})$	$SMT(\varphi_h^{\mathcal{T}} \cup \psi_{s,i}^{\mathcal{T}} \cup \Theta_i^{\mathcal{T}})$
0	$\{\}$	$\{C_0, C_1, C_2, C_3\}$	15	UNSAT
1	$\{\theta_1, \theta_2, \theta_5\}$	$\{\ \ ,\ \ , C_2, C_3\}$	8	SAT

Outline

OMT($\mathcal{LIRA} \cup \mathcal{T}$) captures lots of interesting problems

	Boolean formulas	Sets of $\mathcal{LIRA}$ constraints	SMT($\mathcal{LIRA}$)	SMT($\mathcal{LIRA} \cup \bigcup_i \mathcal{T}_i$)
DECISION (Satisfiability)				
OPTIMIZATION with PB cost function and constraints				
OPTIMIZATION with linear cost function				

OMT($\mathcal{LIRA} \cup \mathcal{T}$) captures lots of interesting problems

	Boolean formulas	Sets of $\mathcal{LIRA}$ constraints	SMT($\mathcal{LIRA}$)	SMT($\mathcal{LIRA} \cup \bigcup_i \mathcal{T}_i$)
DECISION (Satisfiability)				SMT($\mathcal{T}$)
OPTIMIZATION with PB cost function and constraints				
OPTIMIZATION with linear cost function				

OMT($\mathcal{LIRA} \cup \mathcal{T}$) captures lots of interesting problems

	Boolean formulas	Sets of $\mathcal{LIRA}$ constraints	SMT($\mathcal{LIRA}$)	SMT($\mathcal{LIRA} \cup \bigcup_i \mathcal{T}_i$)
DECISION (Satisfiability)				
OPTIMIZATION with PB cost function and constraints	(Weighted) MaxSAT PB Opt.			
OPTIMIZATION with linear cost function				

OMT($\mathcal{LIRA} \cup \mathcal{T}$) captures lots of interesting problems

	Boolean formulas	Sets of $\mathcal{LIRA}$ constraints	SMT($\mathcal{LIRA}$)	SMT($\mathcal{LIRA} \cup \bigcup_i \mathcal{T}_i$)
DECISION (Satisfiability)				
OPTIMIZATION with PB cost function and constraints		MaxSMT and SMT($\mathcal{T}$) with PB cost funct.		
OPTIMIZATION with linear cost function				

OMT($\mathcal{LIRA} \cup \mathcal{T}$) captures lots of interesting problems

	Boolean formulas	Sets of $\mathcal{LIRA}$ constraints	SMT($\mathcal{LIRA}$)	SMT($\mathcal{LIRA} \cup \bigcup_i \mathcal{T}_i$)
DECISION (Satisfiability)		LP		
OPTIMIZATION with PB cost function and constraints				
OPTIMIZATION with linear cost function				

OMT($\mathcal{LIRA} \cup \mathcal{T}$) captures lots of interesting problems

	Boolean formulas	Sets of $\mathcal{LIRA}$ constraints	SMT($\mathcal{LIRA}$)	SMT($\mathcal{LIRA} \cup \bigcup_i \mathcal{T}_i$)
DECISION (Satisfiability)			ILP, MILP, DP, LGDP	
OPTIMIZATION with PB cost function and constraints				
OPTIMIZATION with linear cost function				

OMT($\mathcal{LIRA} \cup \mathcal{T}$) captures lots of interesting problems

	Boolean formulas	Sets of $\mathcal{LIRA}$ constraints	SMT($\mathcal{LIRA}$)	SMT($\mathcal{LIRA} \cup \bigcup_i \mathcal{T}_i$)
DECISION (Satisfiability)				
OPTIMIZATION with PB cost function and constraints				
OPTIMIZATION with linear cost function				

OMT($\mathcal{LIRA} \cup \mathcal{T}$)

OMT($\mathcal{LIRA} \cup \mathcal{T}$) captures lots of interesting problems

	Boolean formulas	Sets of $\mathcal{LIRA}$ constraints	SMT($\mathcal{LIRA}$)	SMT($\mathcal{LIRA} \cup \bigcup_i \mathcal{T}_i$)
DECISION (Satisfiability)		LP	ILP, MILP, DP, LGDP	SMT($\mathcal{T}$)
OPTIMIZATION with PB cost function and constraints	(Weighted) MaxSAT PB Opt.	MaxSMT and SMT($\mathcal{T}$) with PB cost funct.		
OPTIMIZATION with linear cost function		OMT($\mathcal{LIRA} \cup \mathcal{T}$)		

(Finite Domain) Constraint Programming

back

FDCP/MILP

- Very efficient on (integer) linear arithmetic / combinatorial reasoning
- Very efficient handling of global constraints (e.g. all-different)
- Booleans typically represented as 0-1 integers
- (typically) finite precision arithmetic

SMT/OMT

- Very efficient on Boolean reasoning
- Supports other theories (*Array, Bit-Vectors, Strings, ...*)
- Incremental
- infinite precision arithmetic
- Other functionalities: all-smt, proofs, unsat-cores, interpolants, ...

Some OMT tools

- BCLT [44, 35]
  `http://www.cs.upc.edu/~oliveras/bclt-main.html`
- OPTIMATHSAT [52, 53, 55, 54, 57], on top of MATHSAT [22]
  `http://optimathsat.disi.unitn.it`
- SYMBA [38], on top of Z3 [24]
  `https://bitbucket.org/arieg/symba/src`
- Z3 [16, 15], on top of Z3 [24]
  `http://z3.codeplex.com`

More Recently:

- HAZEL [40]. $\Longrightarrow$ *$\mathcal{BV}$, incremental*
- CEGIO [7, 9] $\Longrightarrow$ *counterexample guided inductive optimization*
- MAXHS-MSAT [27] $\Longrightarrow$ *MaxSMT with* Implicit Hitting Set (IHS) algorithm
- PULI [33]. $\Longrightarrow$ *$\mathcal{LIA}$ cost functions, (based on* linear regression*)*

OMT Applications (OPTIMATHSAT)

Real-Time Systems. Worst-Case Execution Time (WCET) of programs [28]
$\Longrightarrow$ reproduced with OPTIMATHSAT [3]

Requirements Engineering. Constrained Goal Models with resources, preferences and goals [41, 42, 43].
$\Longrightarrow$ OPTIMATHSAT backend engine of CGM-TOOL [1]

Machine Learning. Inference & Learning in Hybrid domains [46, 60].
$\Longrightarrow$ OPTIMATHSAT backend engine of LMT tool [2]

Quantum Annealing. Solving SAT and MaxSAT with D-Wave 2000Q QAs [12, 13]
$\Longrightarrow$ offline used of OPTIMATHSAT to generate optimal QUBO encodings of Boolean functions

Formal Verification & Model Checking. Synthesis of Barrier Certificates for Hybrid Dynamical Systems [48]
$\Longrightarrow$ OPTIMATHSAT used as oracle to separate safe/unsafe regions starting from a simulation

Scheduling. Optimal sleep/wake-up scheduling for WSNs [32, 34, 33]
$\Longrightarrow$ OPTIMATHSAT used to deal with increasingly denser WSNs [34]

OMT Applications (Other tools)

Static Analysis.

- Generation of Invariants and Proving Termination via *Constraint-based* method [19]
- Finding Inductive Invariants via *Local Policy Iteration* [30, 31]

Formal Verification & Model Checking.

- Computing Loop Iterations for Bounded Program Verification [39]

Scheduling and Planning with Resources.

- Optimal plans for multi-robot systems [36, 37]
- Task planning for smart factories [14]
- Optimal Job-Shop Scheduling with OMT [50]
- Synthesis Communication Schedules for Time Sensitive Networks [45]

Software Security Engineering.

- Multi-Objective Workflow Satisfiability Problem [11]

Outline

Ongoing Work & Research Directions on OMT

Field still far from maturity, lots of possible research directions:

- Improve efficiency!
- OMT on different theories, e.g.:
  - Bit vectors ([16, 40])
  - $\mathcal{NLA}(\mathbb{R})$
  - $\mathcal{NLA}(\mathbb{Z})$ ([35])
  - Floating point ([61])
- Exploit alternative SMT schemas (e.g., Model-Construction SMT)
- Hybrid techniques, integration with techniques in neighbour fields (MaxSAT, PB, CSP, MILP, CA, ...)
- Extensive empirical comparison wrt. techniques in neighbour fields
  (MaxSAT, PB, CSP, MILP, ...)
- Bridge SMT/OMT with CSP/COP (Minizinc)

To this extent....

Announcement

PHD POSITION available in Trento on
"Advancing Optimization Modulo Theories"
The call will expire in a couple of months.

Please contact me if interested: roberto.sebastiani@unitn.it.
(Se also flier on the desk.)


"That's all Folks!"

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Outline

Outline

Solving OMT($\mathcal{LRA}$) [52, 53]

General idea

Combine standard SMT and LP minimization techniques.

Offline Schema

SMT solver and LP minimizer used as blackbox procedures.

$\Longrightarrow$ no need to hack the code of the SMT solver

Inline Schema

Search for minimum integrated inside the CDCL loop of the SMT solver.

Solving OMT($\mathcal{LRA}$) [52, 53]

General idea

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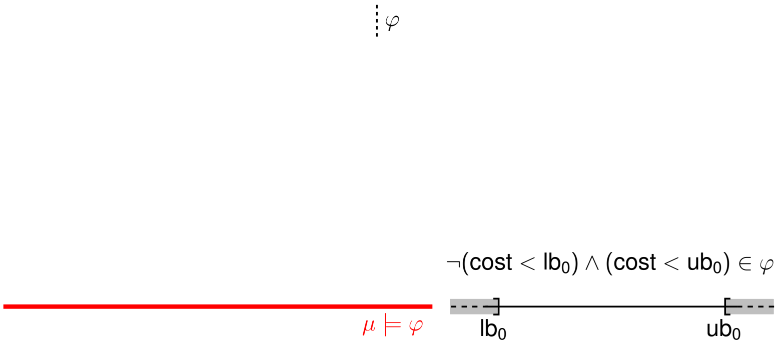
Inline Schema

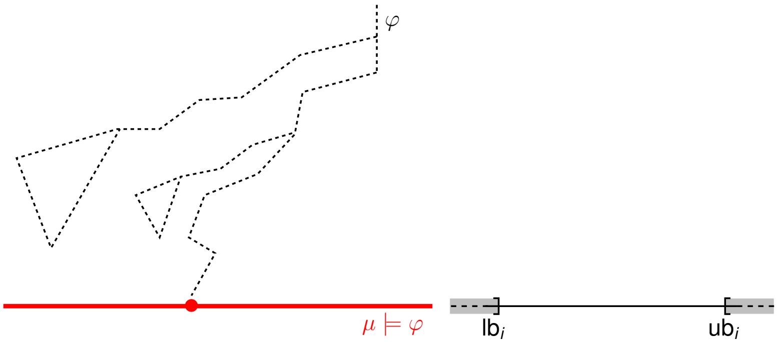
Search for minimum integrated inside the CDCL loop of the SMT solver.

Inline Version: Linear-Search Strategy



- Search for optimum integrated inside CDCL search schema
- Minimizer called incrementally (no restarting of $\mathcal{LRA}$-solver)
- Learned clauses drive backjumping up to level 0
- Intermediate-assignment $\mathcal{LRA}$-checking (early-pruning) plays the role of “bounding” in a Branch & Bound fashion

Inline Version: Linear-Search Strategy



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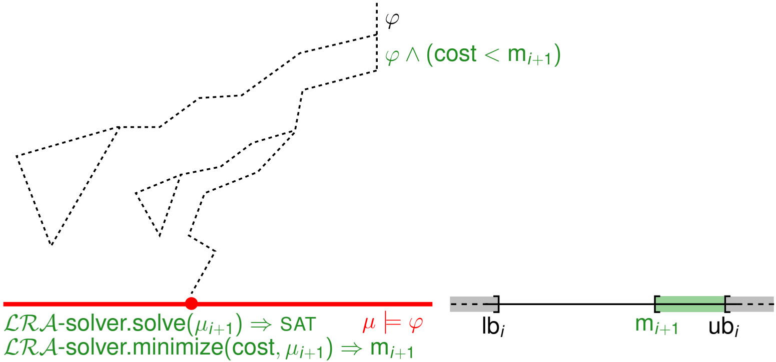

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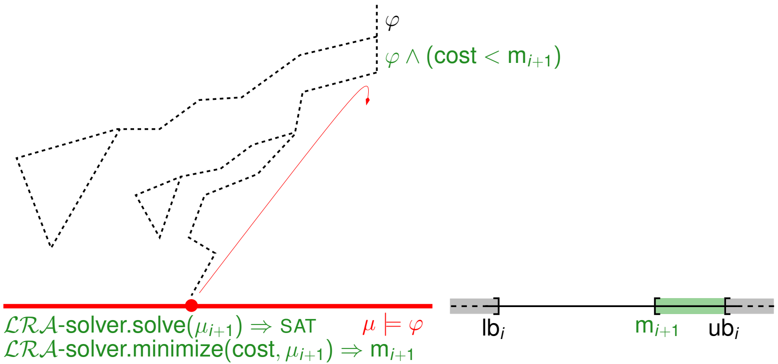

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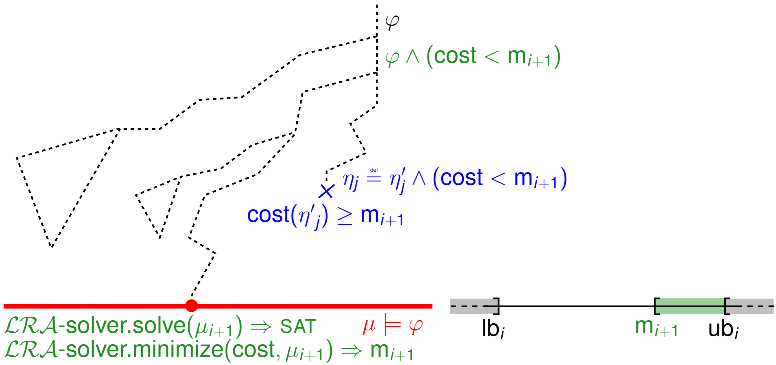

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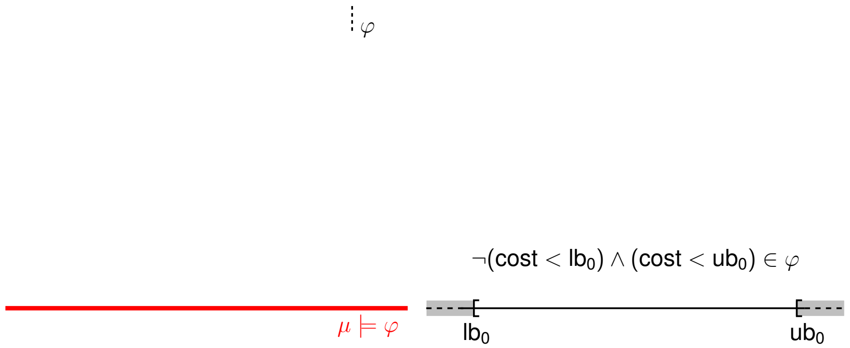
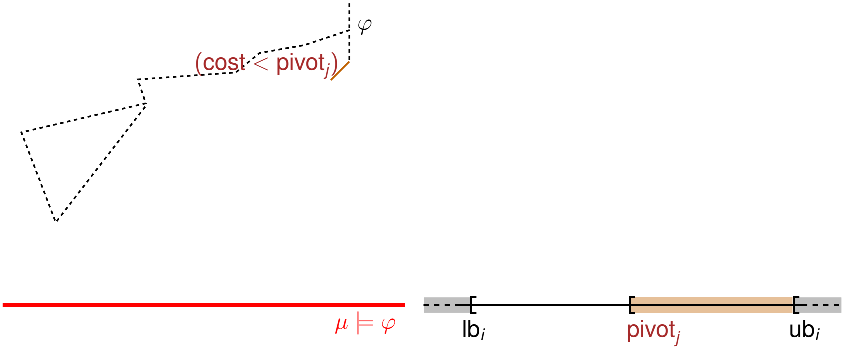
Inline Version: Binary-Search Strategy



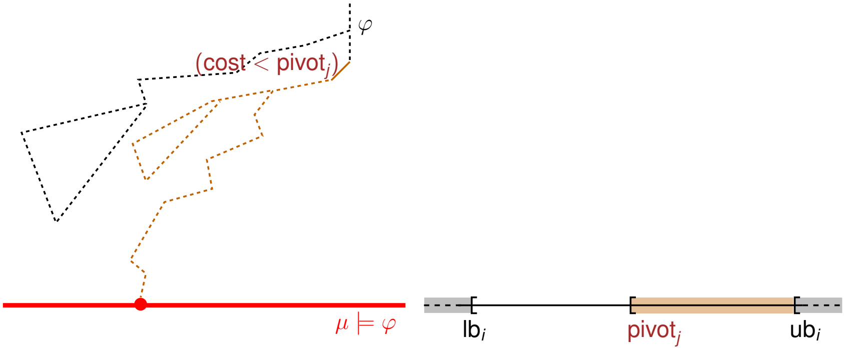
- Range-minimization loop embedded within CDCL search schema
- Level 0: update pivot_j and decide $(\text{cost} < \text{pivot}_j)$

Inline Version: Binary-Search Strategy



- Range-minimization loop embedded within CDCL search schema
- Level 0: update pivot$_j$ and decide (cost $<$ pivot$_j$)

Inline Version: Binary-Search Strategy



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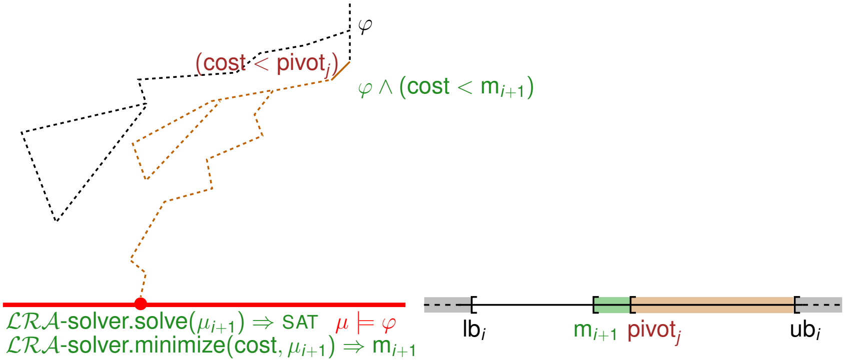
Inline Version: Binary-Search Strategy



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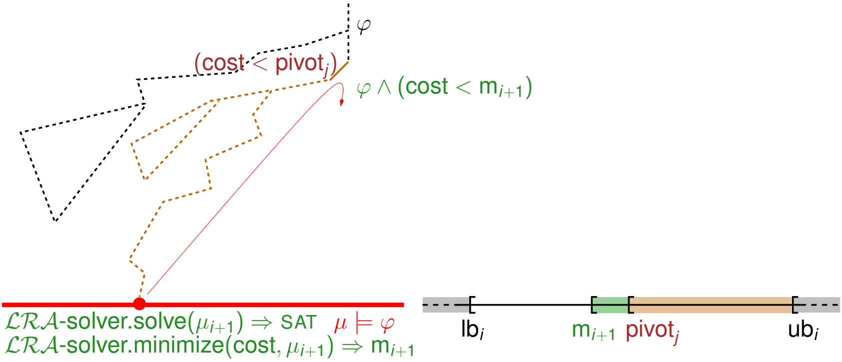

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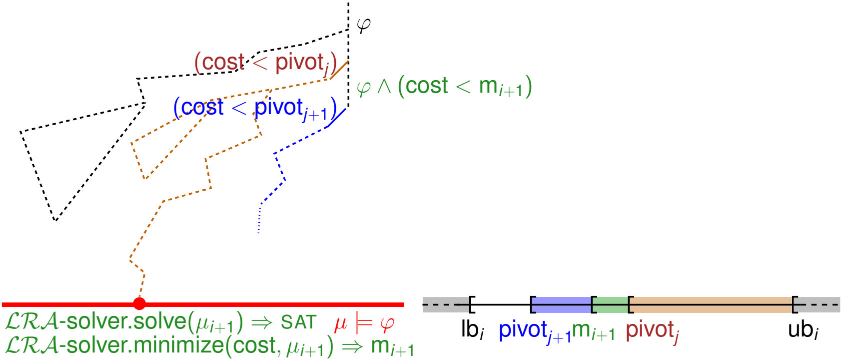

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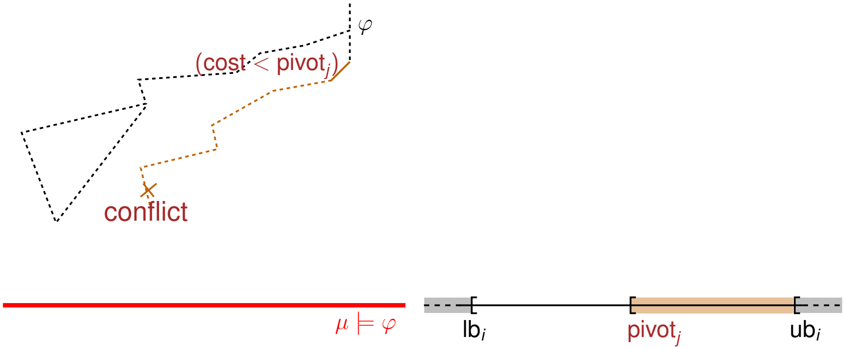

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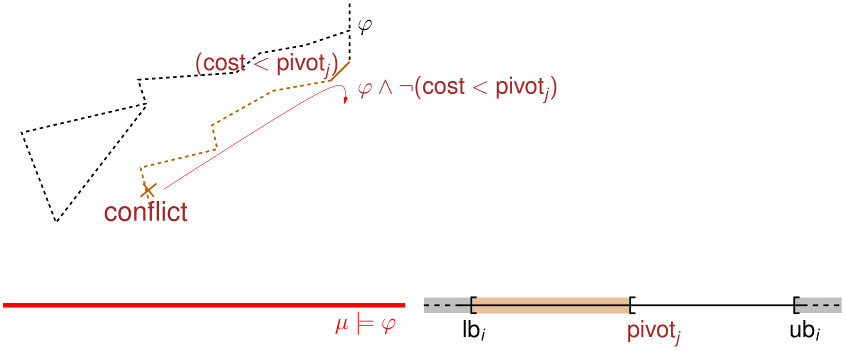
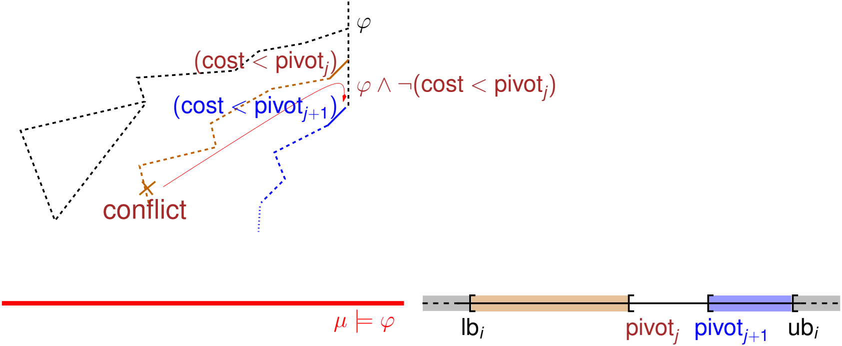


- Range-minimization loop embedded within CDCL search schema
- Level 0: update pivot$_j$ and decide (cost < pivot$_j$)

Inline Version: Binary-Search Strategy



- Range-minimization loop embedded within CDCL search schema
- Level 0: update pivot_j and decide $(\text{cost} < \text{pivot}_j)$

Outline

OMT($\mathcal{BV}$)

back

Minimization of an unsigned Bit-Vector

Given a pair $\langle\varphi, \text{cost}\rangle$, where cost $\stackrel{\text{def}}{=}$ [cost[0], ..., cost[$n-1$]] is an **unsigned** $\mathcal{BV}$ of n bits:

- **Reduction to:**
  - Lexicographic OMT: $\langle\varphi, \{\text{cost}[0] \neq 0, ..., \text{cost}[n-1] \neq 0\}\rangle_{\mathcal{L}}$
  - MaxSMT [16, 17]: $\langle\varphi, \bigcup_{i=0}^{i=n-1}\langle\text{cost}[i] \neq 0, 1\rangle\rangle$
- **OMT-based Approach:** linear-search, binary-search and adaptive-search
- **Ad-Hoc Algorithms:**
  - OBV-WA [40]
    - each cost[i] transformed into a *high-priority* decision variable
    - the *phase-saving* of each cost[i] initialized to 0
  - OBV-BS [40]
    - binary search over the bits [cost[0], ..., cost[$n-1$]]
    - at most *n incremental* calls to the underlying SMT solver

Question:
How to deal with other $\mathcal{BV}$ goals?

- signed vs. unsigned
- maximization vs minimization

OMT($\mathcal{BV}$) - Signed/Unsigned $\mathcal{BV}$ [61]

Example: encoding of a 8-bits Bit-Vector

Unsigned:

0	1	2	3	4	5	6	7		
1	**1**	**1**	**1**	**1**	**1**	**1**	**1**	255	Positive
1	1	1	1	1	1	1	0	254	
...									
1	0	0	0	0	0	0	1	129	
1	0	0	0	0	0	0	0	128	
0	1	1	1	1	1	1	1	127	
0	1	1	1	1	1	1	0	126	
...									
0	0	0	0	0	0	0	1	1	
0	**0**	**0**	**0**	**0**	**0**	**0**	**0**	0	

Signed: (Two's complement)

0	1	2	3	4	5	6	7		
0	**1**	**1**	**1**	**1**	**1**	**1**	**1**	127	Positive
0	1	1	1	1	1	1	0	126	
...									
0	0	0	0	0	0	0	1	1	
0	0	0	0	0	0	0	0	0	
1	1	1	1	1	1	1	1	-1	Negative
1	1	1	1	1	1	1	0	-2	
...									
1	0	0	0	0	0	0	1	-127	
1	**0**	**0**	**0**	**0**	**0**	**0**	**0**	-128	

Attractor *attr* **for** cost**:** when minimizing, it's the **smallest** $\mathcal{BV}$-value of the same sort of cost.

- it's the ideal result of the optimization search
- depends on signed/unsigned

[Dual for Maximization]

OMT($\mathcal{BV}$) - Signed/Unsigned $\mathcal{BV}$ [61]

Reduction to unsigned $\mathcal{BV}$ (minimization)

Given an *attractor attr* for cost, both $\mathcal{BV}$s of n bits, replace cost with

$$\text{cost } \mathbf{xor}_n \ attr$$

Example: maximization of a signed 8-bits Bit-Vector

Before: cost

0	1	2	3	4	5	6	7		
0	**1**	**1**	**1**	**1**	**1**	**1**	**1**	127	Positive
0	1	1	1	1	1	1	0	126	
...									
0	0	0	0	0	0	0	1	1	
0	0	0	0	0	0	0	0	0	
1	1	1	1	1	1	1	1	−1	Negative
1	1	1	1	1	1	1	0	−2	
...									
1	0	0	0	0	0	0	1	−127	
1	0	0	0	0	0	0	0	−128	

After: cost $\mathbf{xor}_8$ #*b*0111111

0	1	2	3	4	5	6	7		
0	**0**	**0**	**0**	**0**	**0**	**0**	**0**	127	Positive
0	0	0	0	0	0	0	1	126	
...									
0	1	1	1	1	1	1	0	1	
0	1	1	1	1	1	1	1	0	
1	0	0	0	0	0	0	0	−1	Negative
1	0	0	0	0	0	0	1	−2	
...									
1	1	1	1	1	1	1	0	−127	
1	1	1	1	1	1	1	1	−128	

OMT($\mathcal{FP}$) [61]

back

Goal: find a model $\mathcal{M}$ of φ for which the value of cost is minimum.

Simplification: $\exists\mathcal{M}$ s.t. $\mathcal{M} \models \varphi$ and $\mathcal{M}(\text{cost}) \neq$ NaN.
$\Longrightarrow$ replace φ with $\varphi \wedge \text{cost} \neq$ NaN

$\mathcal{FP}$ Minimization Approaches

- **Reduction to Bit-Vector Optimization:**
  - $\mathcal{BV}$ and $\mathcal{FP}$ are not Nelson-Oppen disjoint!
    $\Longrightarrow$ can only use eager $\mathcal{BV}/\mathcal{FP}$ SMT-solving approach
- **OMT-based Approach:** linear-search, binary-search and adaptive-search
- **Ad-Hoc Algorithms:**
  - OFP-BS (based on OBV-BS [40])
    - binary search over the bits [cost[0], ..., cost[$n-1$]]
    - at most *n incremental* calls to the underlying SMT solver

OMT($\mathcal{FP}$) [61]

Example: Encoding of a $\mathcal{FP}_{\langle 3,5 \rangle}$

0	1	2	3	4	5	6	7		
0	1	1	1	0	0	0	0	$+\infty$	
0	1	1	0	1	1	1	1	$^{31}/_{2}$	
				…					Positive
0	0	0	0	0	0	0	1	$^{1}/_{64}$	
0	0	0	0	0	0	0	0	$+0$	
1	0	0	0	0	0	0	0	-0	
1	0	0	0	0	0	0	1	$-^{1}/_{64}$	
				…					Negative
1	1	1	0	1	1	1	1	$-^{31}/_{2}$	
1	1	1	1	0	0	0	0	$-\infty$	

Minimization in the

- **Positive Domain**, go towards

0	0	0	0	0	0	0	0	$+0$

- **Negative Domain**, go towards

1	1	1	1	1	1	1	1	NaN

- **unless** the exponent is all **1s**, then go towards

?	1	1	1	0	0	0	0	$+\infty/-\infty$

Dynamic Attractor $attr_{\tau_k}$ **for** cost**:** given an assignment τ_k to the first k bits of cost, it's the **smallest** $\mathcal{FP}$-value different from NaN s.t.

$$\forall_{i=0}^{i=k-1} attr_{\tau_k}[i] = \tau_k[i]$$

- The ideal result of the optimization wrt. **current** search horizon

OMT($\mathcal{FP}$) - OFP-BS [61]

Idea: Use $attr_{\tau_k}$ as look-ahead.

- **if** $(\mathcal{M}(\text{cost}[k]) \neq attr_{\tau_k}[k])$ **then**
  SMT.INCREMENTAL_CHECK($\varphi \wedge \tau_k \wedge \text{cost}[k] = attr_{\tau_k}[k]$) // try improve cost
  - UNSAT $\Longrightarrow$ update τ_k and $attr_{\tau_k}$
  - SAT $\Longrightarrow$ update τ_k and $\mathcal{M}$
- otherwise: **skip**

Disclosure: based on OBV-BS [40].

Example: minimization of a $\mathcal{FP}_{\langle 3,5 \rangle}$

k	$\mathcal{M}$(cost)		$\tau_\mathbf{k}$	**attr**$_{\tau_\mathbf{k}}$		
0	0 1 1 0 1 1 1 1	$^{31}/_{2}$		1 1 1 1 0 0 0 0	$-\infty$	$\Longrightarrow$ UNSAT
1	0 1 1 0 1 1 1 1	$^{31}/_{2}$	0	0 0 0 0 0 0 0 0	$+0$	$\Longrightarrow$ SAT
2	0 0 0 0 0 0 1 0	$^{1}/_{32}$	0 0	0 0 0 0 0 0 0 0	$+0$	$\Longrightarrow$ **skip**
6	0 0 0 0 0 0 1 0	$^{1}/_{32}$	0 0 0 0 0 0	0 0 0 0 0 0 0 0	$+0$	$\Longrightarrow$ UNSAT
7	0 0 0 0 0 0 1 0	$^{1}/_{32}$	0 0 0 0 0 0 1	0 0 0 0 0 0 1 0	$^{1}/_{32}$	$\Longrightarrow$ **skip**
8	0 0 0 0 0 0 1 0	$^{1}/_{32}$	0 0 0 0 0 0 1 0	0 0 0 0 0 0 1 0	$^{1}/_{32}$	$\Longrightarrow$ **end.**

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2	0 0 0 0 0 0 1 0	$^{1}/_{32}$	0 0	0 0 0 0 0 0 0 0	$+0$	$\Longrightarrow$ skip
6	0 0 0 0 0 0 1 0	$^{1}/_{32}$	0 0 0 0 0 0	0 0 0 0 0 0 0 0	$+0$	$\Longrightarrow$ UNSAT
7	0 0 0 0 0 0 1 0	$^{1}/_{32}$	0 0 0 0 0 0 1	0 0 0 0 0 0 1 0	$^{1}/_{32}$	$\Longrightarrow$ skip
8	0 0 0 0 0 0 1 0	$^{1}/_{32}$	0 0 0 0 0 0 1 0	0 0 0 0 0 0 1 0	$^{1}/_{32}$	$\Longrightarrow$ end.

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k	$\mathcal{M}$(cost)		τ_k	$attr_{\tau_k}$		
0	0 1 1 0 1 1 1 1	$^{31}/_{2}$		1 1 1 1 0 0 0 0	$-\infty$	$\Longrightarrow$ UNSAT
1	0 1 1 0 1 1 1 1	$^{31}/_{2}$	0	0 0 0 0 0 0 0 0	$+0$	$\Longrightarrow$ SAT
2	0 0 0 0 0 0 1 0	$^{1}/_{32}$	0 0	0 0 0 0 0 0 0 0	$+0$	$\Longrightarrow$ skip
6	0 0 0 0 0 0 1 0	$^{1}/_{32}$	0 0 0 0 0 0	0 0 0 0 0 0 0 0	$+0$	$\Longrightarrow$ UNSAT
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1	0 1 1 0 1 1 1 1	$^{31}/_{2}$	0	0 0 0 0 0 0 0 0	$+0$	$\Longrightarrow$ SAT
2	0 0 0 0 0 0 1 0	$^{1}/_{32}$	0 0	0 0 0 0 0 0 0 0	$+0$	$\Longrightarrow$ **skip**
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7	0 0 0 0 0 0 1 0	$^{1}/_{32}$	0 0 0 0 0 0 1	0 0 0 0 0 0 1 0	$^{1}/_{32}$	$\Longrightarrow$ **skip**
8	0 0 0 0 0 0 1 0	$^{1}/_{32}$	0 0 0 0 0 0 1 0	0 0 0 0 0 0 1 0	$^{1}/_{32}$	$\Longrightarrow$ **end.**

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2	0 0 0 0 0 0 1 0	$^{1}/_{32}$	0 0	0 0 0 0 0 0 0 0	$+0$	$\Longrightarrow$ **skip**
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7	0 0 0 0 0 0 1 0	$^{1}/_{32}$	0 0 0 0 0 0 1	0 0 0 0 0 0 1 0	$^{1}/_{32}$	$\Longrightarrow$ skip
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8	0 0 0 0 0 0 1 0	$^{1}/_{32}$	0 0 0 0 0 0 1 0	0 0 0 0 0 0 1 0	$^{1}/_{32}$	$\Longrightarrow$ **end.**

Outline

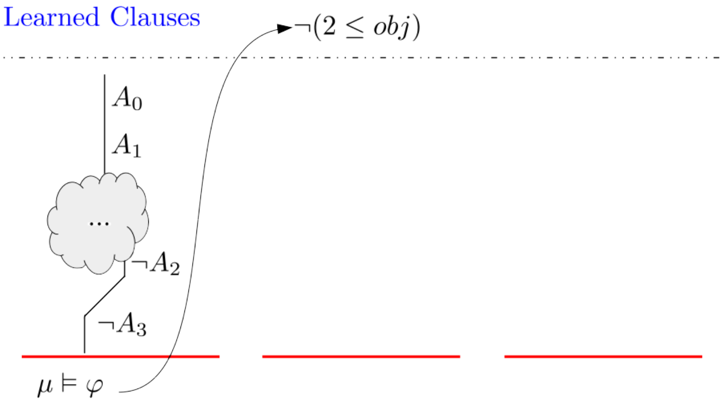
Running Example: performance bottleneck

back

Problem:

- $\langle \varphi, \texttt{min}(\text{cost}) \rangle$, where cost $:= w \cdot \sum_{i=0}^{n-1} A_i$, currently $obj = k \cdot w$
- OPTIMIZATION STEP: learn $\neg(k \cdot w \leq \text{cost})$ and restart/jump to *level* 0

Example: with $k = 2$, $w = 1$ and $n = 4$



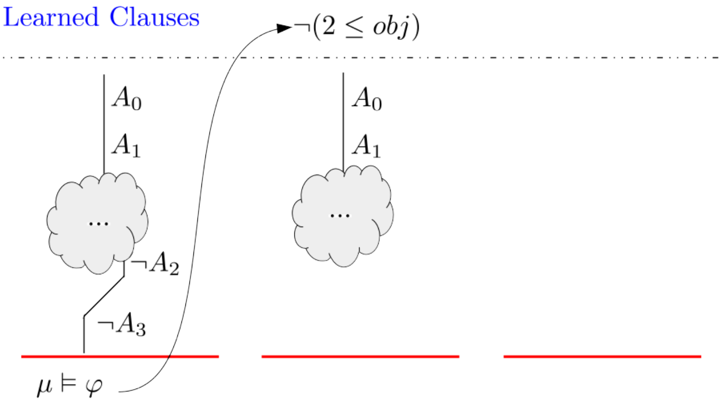
Running Example: performance bottleneck

back

Problem:

- $\neg(k \leq \text{cost})$ causes the inconsistency of $\binom{n}{k}$ truth assignments satisfying exactly k variables in $A_0, ..., A_{n-1}$

Example: with $k = 2$, $w = 1$ and $n = 4$



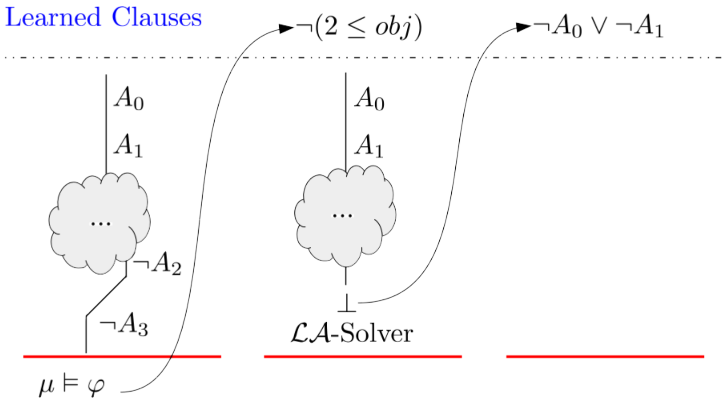
Running Example: performance bottleneck

back

Problem:

- $\neg(k \leq \text{cost})$ causes the inconsistency of $\binom{n}{k}$ truth assignments satisfying exactly k variables in $A_0, ..., A_{n-1}$
  $\Longrightarrow$ inconsistency is not revealed by Boolean Constraint Propagation

Example: with $k = 2$, $w = 1$ and $n = 4$



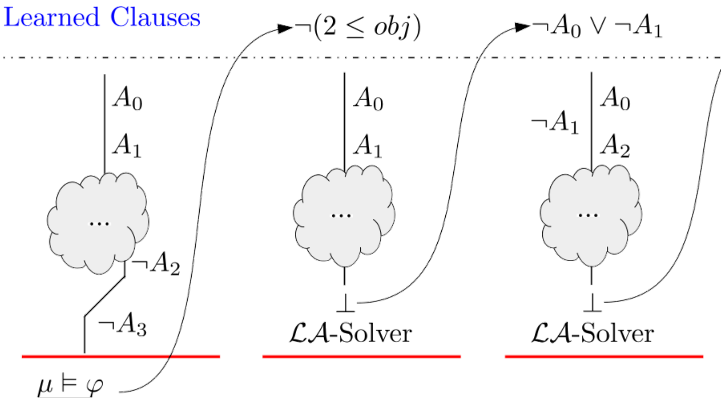
Running Example: performance bottleneck

back

Problem:

- up to $\binom{n}{k}$ (expensive) calls to the $\mathcal{LA}$-Solver required

Example: with $k = 2$, $w = 1$ and $n = 4$



Solution: OMT + sorting networks [56]

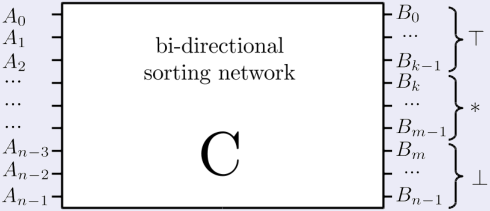
Contribution:

Enriched OMT encoding with bidirectional **sorting networks** [58, 10].

Approach:

Given $\langle \varphi, \text{cost} \rangle$, cost $:= w \cdot \sum_{i=0}^{n-1} A_i$, and a bi-directional **sorting network** relation $C(A_0, ..., A_{n-1}, B_0, ..., B_{n-1})$ s.t.

- k A_i's are $\top \iff \{B_0, ..., B_{k-1}\}$ are $\top$,
- $m - k$ A_i's are $* \iff \{B_k, ..., B_{m-1}\}$ are $*$,
- $n - m$ A_i's are $\bot \iff \{B_m, ..., B_{n-1}\}$ are $\bot$



then we encode it as $\langle \varphi', \text{cost} \rangle$, where

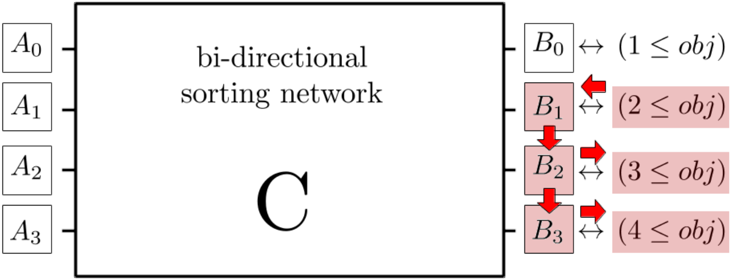
$$\varphi' := \varphi \wedge C(A_0, ..., A_{n-1}, B_0, ..., B_{n-1}) \wedge \bigwedge_{i=0}^{n-1} B_i \leftrightarrow ((i+1) \cdot w \leq \text{cost}) \wedge \bigwedge_{i=0}^{n-2} B_{i+1} \rightarrow B_i$$

Properties: OMT + sorting networks [56]

Properties:

- if $(k \cdot w \leq \text{cost}) = \bot$, then by BCP $\forall i \in [k, n].B_{i-1} = \bot$

Example: with $k = 2$, $w = 1$ and $n = 4$



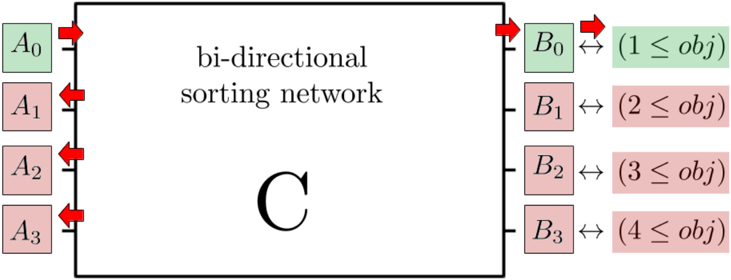
Properties: OMT + sorting networks [56]

Properties:

- if $(k \cdot w \leq \text{cost}) = \perp$, then by BCP $\forall i \in [k, n]. B_{i-1} = \perp$
- as soon as $k-1$ A_i are assigned $\top$
  $\Longrightarrow$ all others are unit-propagated to $\perp$

Dual if $(k \cdot w \leq \text{cost}) = \top$.

Example: with $k = 2$, $w = 1$ and $n = 4$

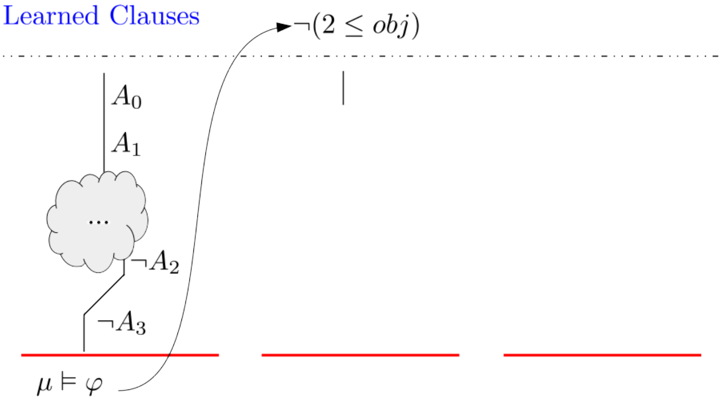


Example: OMT with sorting networks

back

- Optimization Step: learn $\neg(k \cdot w \leq \text{cost})$ and restart/jump to *level* 0

Example: with $k = 2$, $w = 1$ and $n = 4$



Example: OMT with sorting networks

back

- OPTIMIZATION STEP: learn $\neg(k \cdot w \leq \text{cost})$ and restart/jump to *level* 0
- as soon as $k-1$ A_i are assigned $\top$
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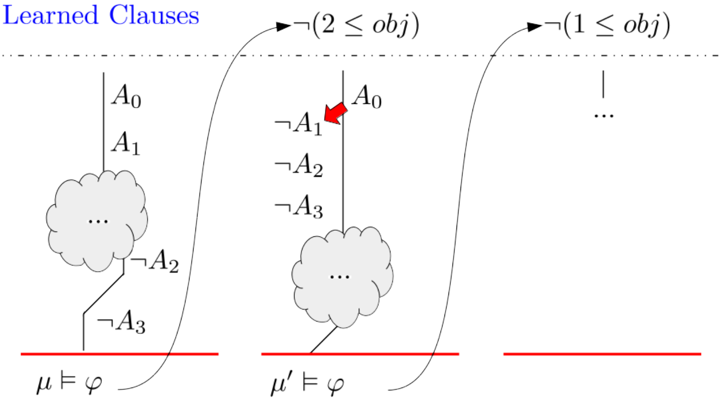
Example: with $k = 2$, $w = 1$ and $n = 4$



Solution: Combine OMT with Sorting Networks

OPTIMATHSAT: sorting networks implemented

- **Bi-directional Sequential Counter** [58], in $O(n^2)$ but incremental
  sum of A_i's, unary representation
- **Bi-directional Cardinality Network** [10, 6], in $O(n\,log^2 n)$
  based on *merge-sort* algorithm

Generalization

The same performance issue occurs for $\langle \varphi, \text{cost} \rangle$, where

$$\text{cost} = \tau_1 + ... + \tau_m,$$

$$\forall_j \in [1, m].\ (\tau_j = w_j \cdot \sum_{i=0}^{i=k_j} A_{ji})\ \wedge\ (0 \leq \tau_j) \wedge (\tau_j \leq w_j \cdot k_j)$$

Solution:

- use a separate sorting circuit for each term τ_j
- add clauses in the form $(w_j \cdot i \leq \tau_j) \rightarrow (w_j \cdot i \leq \text{cost})$

Outline

MaxRes: Maximum Resolution [16]

back

Idea: given a MaxSMT $\langle \varphi_h, \varphi_s \rangle$, treat both φ_h and φ_s as hard clauses.

Analyze conflict τ, where $\tau \stackrel{\text{def}}{=} \tau_h \cup \tau_s$, $\tau_h \subseteq \varphi_h$ and $\tau_s \subseteq \varphi_s$

- if $\tau_s = \emptyset \Longrightarrow$ input problem is unsatisfiable
- else **let** $w_{min} \stackrel{\text{def}}{=} \min(w_i \mid \langle C_i, w_i \rangle \in \tau_s)$ and **relax** the problem:
  - Learn conflict-clause and replace soft-clauses

$$\begin{aligned} \varphi_h \quad &:= \quad \varphi_h \cup \bigvee\nolimits_{\langle C_i, w_i \rangle \in \tau_s} \neg C_i \\ \varphi_s \quad &:= \quad \varphi_s \setminus \tau_s \cup \bigcup\nolimits_{\langle C_i, w_i \rangle \in \tau_s} \langle C_i, w_i - w_{min} \rangle \ \textbf{if} \ \ w_i - w_{min} > 0 \end{aligned}$$

  - if $\mid \tau_s \mid > \ 1 \Longrightarrow$ add compensation clauses

$$\begin{aligned} \varphi_h \quad &:= \quad \varphi_h \cup \bigcup\nolimits_{\langle C_i, w_i \rangle \in \tau_s} . B_i \rightarrow (B_{i-1} \wedge C_i) \\ & \qquad \qquad // \ B_0 := \top, \forall_{i>0}. B_i \text{ is fresh Boolean var} \\ \varphi_s \quad &:= \quad \varphi_s \cup \bigcup\nolimits_{\langle C_i, w_i \rangle \in \{\tau_s \setminus \langle C_1, w_1 \rangle\}} . \langle B_{i-1} \vee C_i, w_{min} \rangle \end{aligned}$$

No Conflict: optimal solution

Outline

Extended SMT-LIBv2 Interface [57]

back

```
(minimize <term> [:id <string>] [:signed]
            [:lower <const_term>] [:upper <const_term>])
(maximize <term> [:id <string>] [:signed]
            [:lower <const_term>] [:upper <const_term>])

(minmax <term> ... <term> [:id <string>] [:signed]
            [:lower <const_term>] [:upper <const_term>])
(maxmin <term> ... <term> [:id <string>] [:signed]
            [:lower <const_term>] [:upper <const_term>])

(assert-soft <term> [:id <string>] [:weight <const_term>])

(check-sat)
(check-allsat (<const_term> ... <const_term>))

(get-objectives)
(load-objective-model <numeral>)
```

Outline

Pareto OMT

back

Definitions:

- A model $\mathcal{M}$ *Pareto-dominates* $\mathcal{M}'$ iff

  $$\forall i.\mathcal{M}(\text{cost}_i) \leq \mathcal{M}'(\text{cost}_i)$$

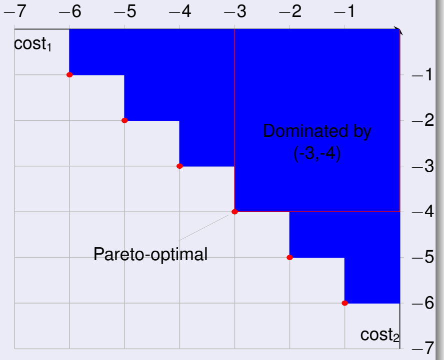
  and

  $$\exists j.\mathcal{M}(\text{cost}_j) < \mathcal{M}'(\text{cost}_j)$$

  (dual for maximization)

- $\mathcal{M}$ is *Pareto-optimal* iff it is not Pareto-dominated by any $\mathcal{M}'$.

Example: $\langle \varphi, \{\text{cost}_1, \text{cost}_2\}\rangle_{\mathcal{P}}$



Goal: given a pair $\langle \varphi, \mathcal{O}\rangle_{\mathcal{P}}$, where $\mathcal{O} \stackrel{\text{def}}{=} \{\text{cost}_1, ..., \text{cost}_N\}$

- find the set of Pareto-optimal models $\{\mathcal{M}_1, ..., \mathcal{M}_M\}$ (i.e. the *Pareto front*)

Pareto OMT: Guided Improvement Algorithm (GIA)

Guided Improvement Algorithm [49, 16]

Given a pair $\langle \varphi, \mathcal{O} \rangle_{\mathcal{P}}$, where $\mathcal{O} \stackrel{\text{def}}{=} \{\text{cost}_1, ..., \text{cost}_N\}$:

- start from random model $\mathcal{M}$ of φ
- **loop:** look for a model $\mathcal{M}'$ of φ that *Pareto-dominates* $\mathcal{M}$
  $\Longrightarrow$ if any, replace $\mathcal{M}$ with $\mathcal{M}'$ and keep looking
- block solutions Pareto-dominated by $\mathcal{M}$
- **repeat**

Infinite Loop:

- some cost_i is unbounded
- some cost_j can always be improved by an *infinitesimal value* (e.g. OMT($\mathcal{LRA}$))

Also: $\mathcal{T}$-minimization procedure not used
$\Longrightarrow$ the same μ may be visited multiple times by CDCL/SAT engine

Pareto OMT: Lexicographic GIA

Observation. If model $\mathcal{M}$ is Lexicographic-optimal for $\langle \varphi, \{\text{cost}_1, ..., \text{cost}_N\} \rangle_{\mathcal{L}}$, then $\mathcal{M}$ is also Pareto-optimal for $\langle \varphi, \{\text{cost}_1, ..., \text{cost}_N\} \rangle_{\mathcal{P}}$.

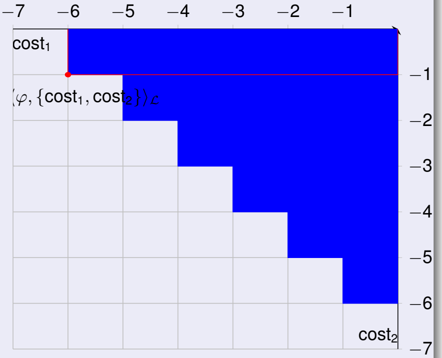
Idea:

- Shuffle $\{\text{cost}_1, ..., \text{cost}_N\}$
  $\Longrightarrow$ *explore from different directions*
- Extract Lexicographic-optimal $\mathcal{M}$
- Learn

  $$\bigvee_{i=1}^{i=N} (\text{cost}_i < \mathcal{M}[\text{cost}_i])$$

  to block Pareto-dominated solutions
- **repeat**

Example: $\langle \varphi, \{\text{cost}_1, \text{cost}_2\} \rangle_{\mathcal{P}}$



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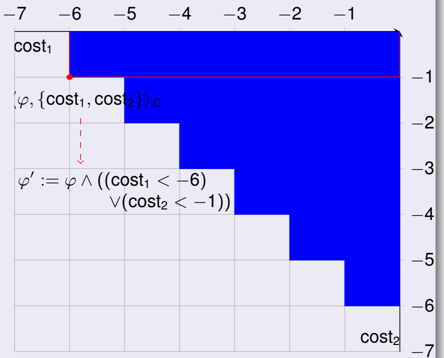

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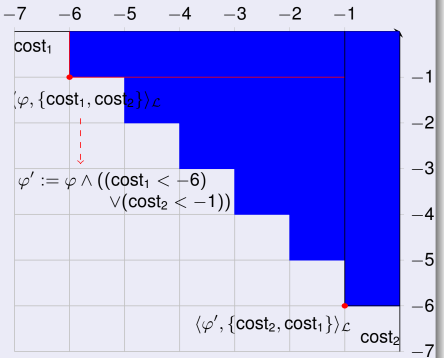

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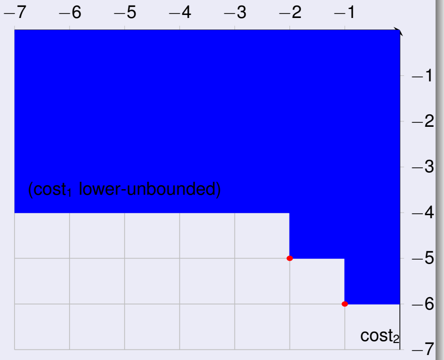
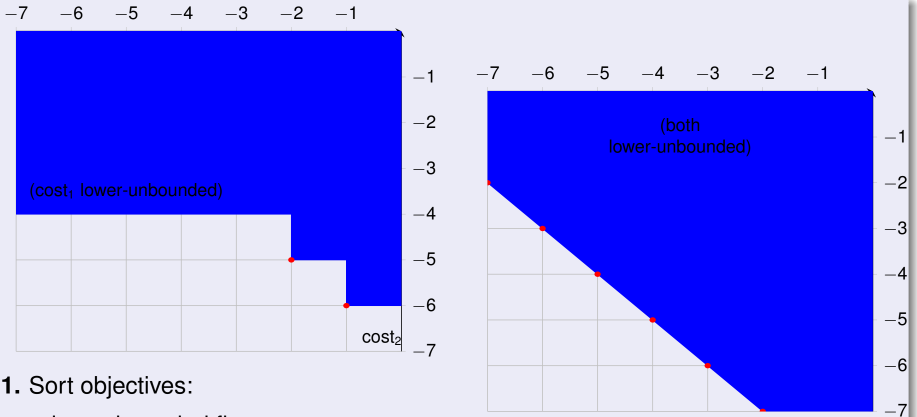


Problem: how to deal with unbounded objectives?

Pareto OMT: dealing with unbounded objectives

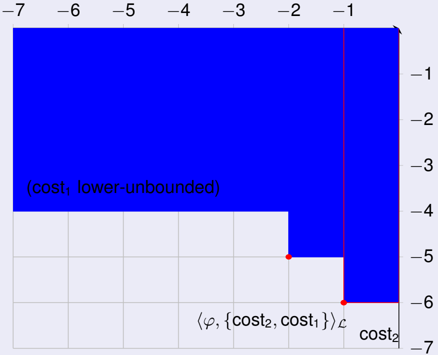


1. Sort objectives:

- lower-bounded first
- lower-unbounded last

before Lex. OMT.

Pareto OMT: dealing with unbounded objectives



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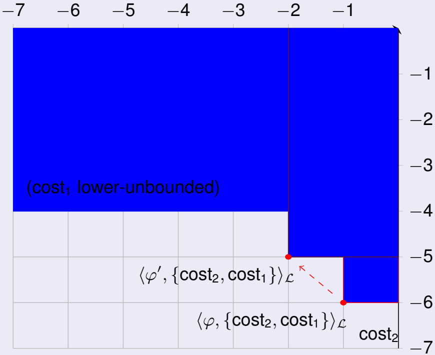
Pareto OMT: dealing with unbounded objectives



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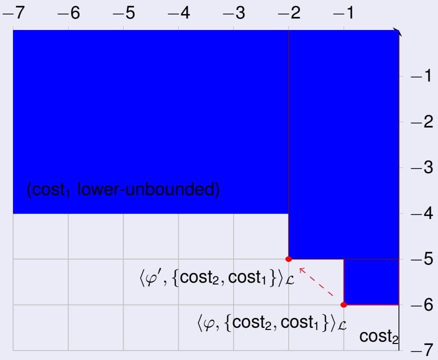

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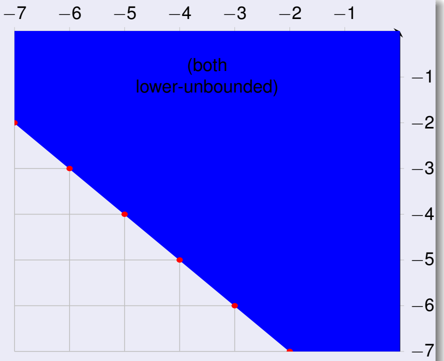
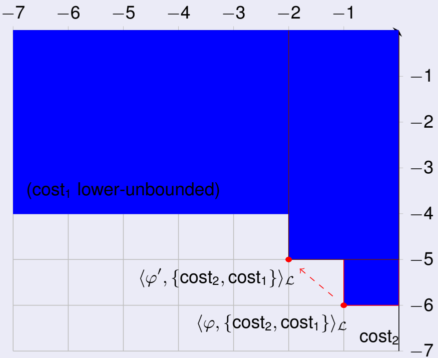


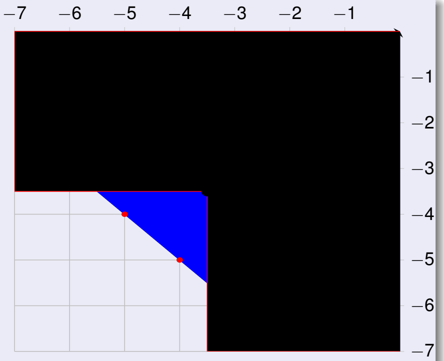
2. If Lex. OMT unbounded, (temporarily) learn:

$$\bigwedge_{i=1}^{i=N} (\text{cost}_i \leq \mathcal{M}[\text{cost}_i])$$

and try again.

Pareto OMT: dealing with unbounded objectives





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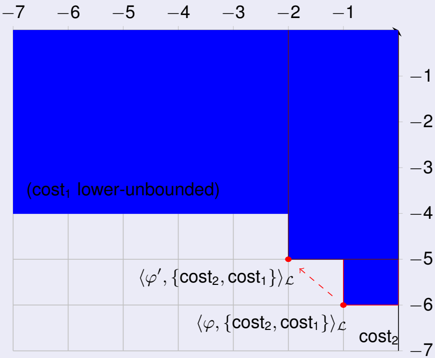
Pareto OMT: dealing with unbounded objectives



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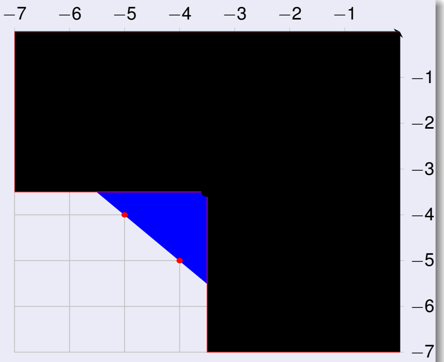

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3. If Lex. OMT still unbounded, give up.