Computing with SAT Oracles

Joao Marques-Silva

SAT/SMT/AR 2019 Summer School

IST, Lisbon, Portugal

July 3-6 2019

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What is SAT?

- SAT is the decision problem for propositional logic
  - Well-formed propositional formulas, with variables, logical connectives: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$, and parenthesis: (,)
  - Often restricted to Conjunctive Normal Form (CNF)

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- SAT is NP-complete [Coo71]

The CDCL SAT disruption

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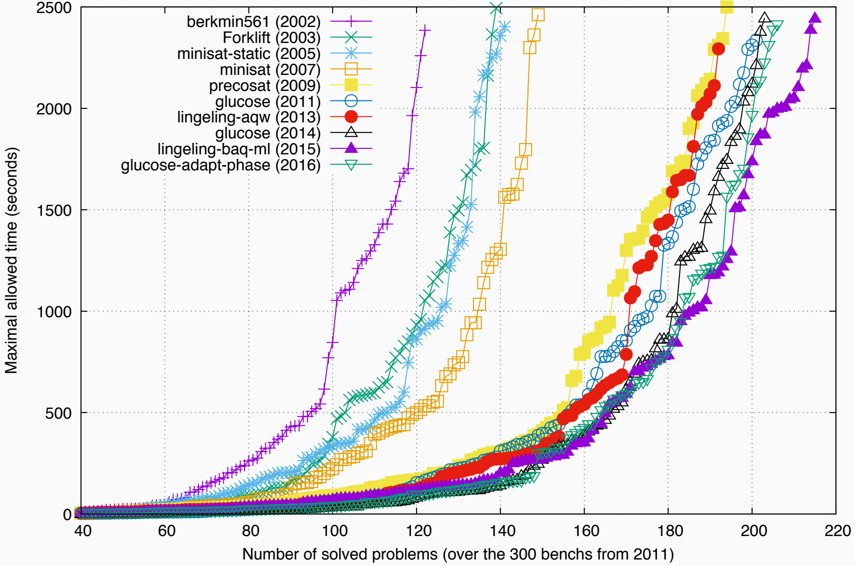
- CDCL SAT solving is a success story of Computer Science
    - Conflict-Driven Clause Learning (CDCL)
    - (CDCL) SAT has impacted many different fields
    - Hundreds (thousands?) of practical applications

CDCL SAT solver (continued) improvement

[Source: Simon 2015]



How good are CDCL SAT solvers?

Demos

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- Sample SAT of solvers:
  1. POSIT: state of the art DPLL SAT solver in 1995
  2. GRASP: first CDCL SAT solver, state of the art 1995∼2000
  3. Minisat: CDCL SAT solver, state of the art until the late 00s
  4. Glucose: modern state of the art CDCL SAT solver
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- Example 1: model checking example (from IBM)
- Example 2: cooperative path finding (CPF)

How good are SAT solvers? – an example

- Cooperative pathfinding (CPF)
  - N agents on some grid/graph
  - Start positions
  - Goal positions
  - Minimize makespan
  - Restricted planning problem

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- Concrete example
  - Gaming grid
  - 1039 vertices
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```
*** tracker: a pathfinding tool ***

Initialization ... CPU Time: 0.004711
Number of variables: 113315
Tentative makespan 1
Number of variables: 226630
Number of assumptions: 1
c Running SAT solver ... CPU Time: 0.718112
c Done running SAT solver ... CPU Time: 0.830099
No solution for makespan 1
Elapsed CPU Time: 0.830112
Tentative makespan 2
Number of variables: 339945
Number of assumptions: 1
c Running SAT solver ... CPU Time: 1.27113
c Done running SAT solver ... CPU Time: 1.27114
No solution for makespan 2
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Tentative makespan 24
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  - Formula w/ 2946190 variables!
- **Note:** In the early 90s, SAT solvers could solve formulas with a few hundred variables!

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Grasping the search space ...

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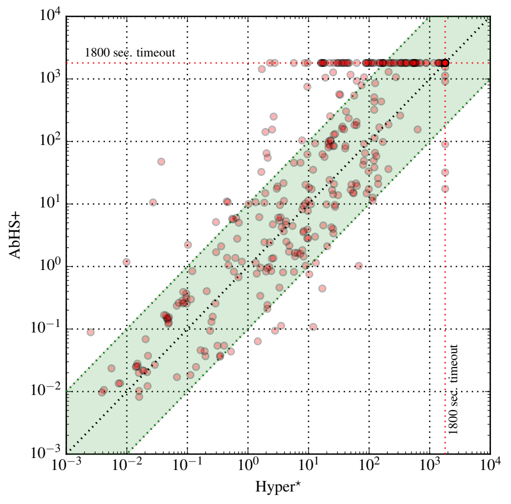
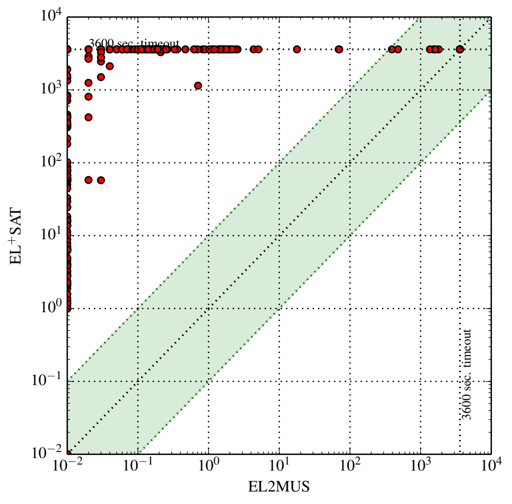
- Search space with 2832875 propositional variables (worst case):
  - # of assignments to $> 2.8 \times 10^{6}$ variables: $\gg 10^{840000}$ **!!**
  - **Obs:** SAT solvers at present (but formula dependent)

SAT can make the difference – propositional abduction



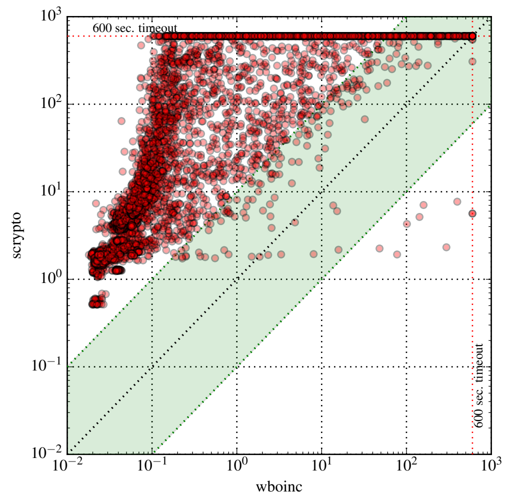
- Propositional abduction instances [IMM16]
  - Implicit hitting set dualization (IHSD)

SAT can make the difference – axiom pinpointing



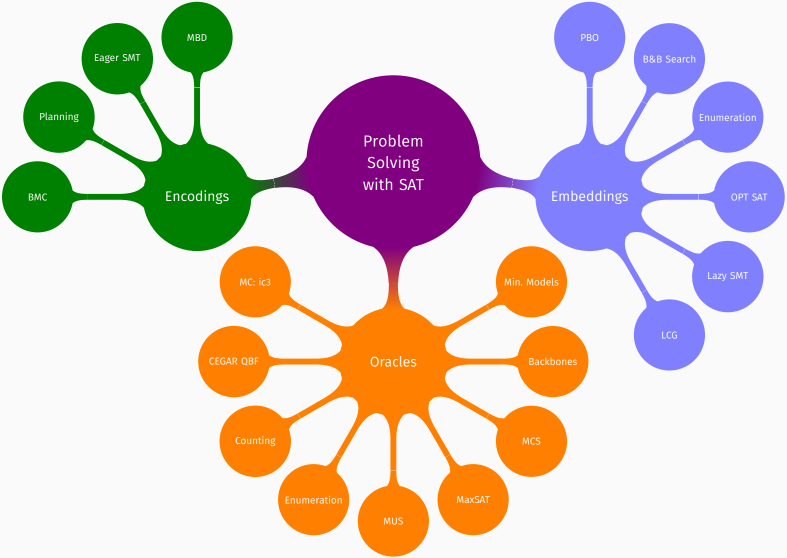
- $\mathcal{EL}^+$ medical ontologies [AMM15]
  - Minimal unsatisfiability (MUSes) & maximal satisfiability (MCSes) & Enumeration

SAT can make the difference – model based diagnosis



- Model-based diagnosis problem instances [MJIM15]
  - Maximum satisfiability (MaxSAT)

CDCL SAT is ubiquitous in problem solving



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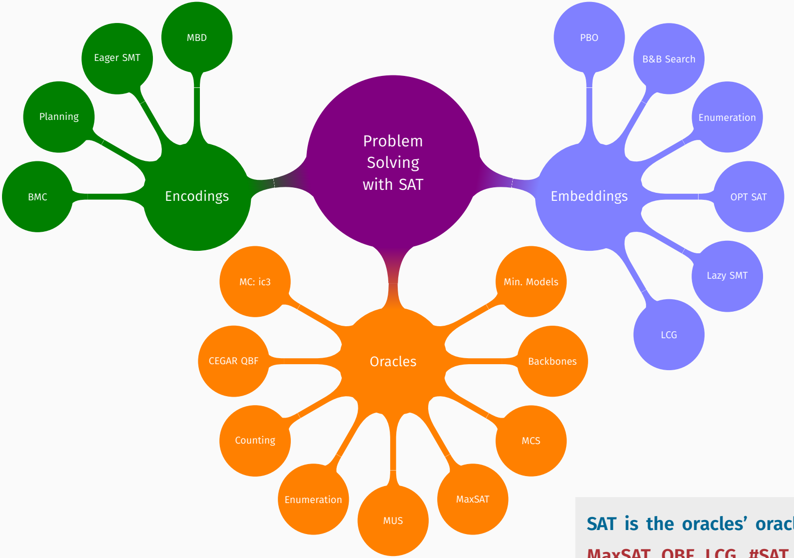

SAT is the oracles' oracle:
MaxSAT, QBF, LCG, #SAT, SMT, ASP, FOL, ...

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  - Minimal Sets over Monotone Predicates (MSMP) — Contact me
  - Enumeration problems
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  - Quantification problems — Contact me
  - (Approximate) counting problems
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- Part #3: Research directions

What this tutorial does not cover ...

- CDCL SAT solvers — A. Biere's talk
  - Clause learning; search restarts; watched literals; VSIDS; ...
- Modeling in propositional logic — Contact me
  - Cardinality constraints; pseudo-boolean constraints; circuits; general constraints; etc.
- Many (high-profile) applications — Contact me
  - Minimal/minimum decision trees/sets [NIPM18, IPNM18]
  - ML model explanations as prime implicants [INMS19]
  - ...

0 Basic Definitions

Preliminaries

- Variables: $w, x, y, z, a, b, c, \ldots$
- Literals: $w, \bar{x}, \bar{y}, a, \ldots$, but also $\neg w, \neg y, \ldots$
- Clauses: disjunction of literals or set of literals
- Formula: conjunction of clauses or set of clauses
- Model (satisfying assignment): partial/total mapping from variables to $\{0, 1\}$ that satisfies formula
- Each clause can be satisfied, falsified, but also unit, unresolved
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$$\mathcal{F} \triangleq (r) \wedge (\bar{r} \vee s) \wedge (\bar{w} \vee a) \wedge (\bar{x} \vee b) \wedge (\bar{y} \vee \bar{z} \vee c) \wedge (\bar{b} \vee \bar{c} \vee d)$$

  - Example models:

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  - Example models:
    - $\{r, s, a, b, c, d\}$
    - $\{r, s, \bar{x}, y, \bar{w}, z, \bar{a}, b, c, d\}$

Resolution

- Resolution rule: [DP60, Rob65]

$$\frac{(\alpha \vee x) \qquad\qquad (\beta \vee \bar{x})}{(\alpha \vee \beta)}$$

  - Complete proof system for propositional logic

Resolution

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$$\frac{(\alpha \lor x) \qquad\qquad (\beta \lor \bar{x})}{(\alpha \lor \beta)}$$

  - Complete proof system for propositional logic

$$\begin{tikzcd}
(x \lor a) \arrow[d, no head] & (\bar{x} \lor a) \arrow[dl, no head] & (\bar{y} \lor \bar{a}) \arrow[dr, no head] & (y \lor \bar{a}) \arrow[d, no head] \\
(a) \arrow[dr, no head] & & & (\bar{a}) \arrow[dll, no head] \\
& \bot & &
\end{tikzcd}$$

  - Extensively used with (CDCL) SAT solvers

Unit propagation

$$\begin{aligned}\mathcal{F} \quad = \quad & (r) \wedge (\bar{r} \vee s) \wedge \\ & (\bar{w} \vee a) \wedge (\bar{x} \vee \bar{a} \vee b) \wedge \\ & (\bar{y} \vee \bar{z} \vee c) \wedge (\bar{b} \vee \bar{c} \vee d)\end{aligned}$$

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- Decisions / Variable Branchings:
  $w = 1, x = 1, y = 1, z = 1$

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- **Unit clause rule**: if clause is unit, its sole literal **must** be satisfied

Unit propagation

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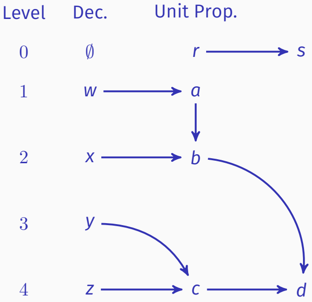
$$



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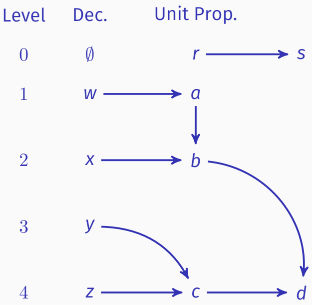

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- Additional definitions:
  - Antecedent (or reason) of an implied assignment
    - $(\bar{b} \vee \bar{c} \vee d)$ for d
  - Associate assignment with decision levels
    - $w = 1 @ 1, x = 1 @ 2, y = 1 @ 3, z = 1 @ 4$
    - $r = 1 @ 0, d = 1 @ 4, \ldots$

Resolution proofs

- Refutation of unsatisfiable formula by iterated resolution operations produces resolution proof
- An example:
  $\mathcal{F} = (\bar{c}) \wedge (\bar{b}) \wedge (\bar{a} \vee c) \wedge (a \vee b) \wedge (a \vee \bar{d}) \wedge (\bar{a} \vee \bar{d})$
- Resolution proof:

$$\begin{tikzcd}[column sep=small]
 & & & (a \vee b) \arrow[dr, no head] & & (\bar{a} \vee c) \arrow[dl, no head] \\
 & & (\bar{c}) \arrow[dr, no head] & & (b \vee c) \arrow[dl, no head] & \\
 & (\bar{b}) \arrow[dr, no head] & & (b) \arrow[dl, no head] & & \\
 & & \bot & & &
\end{tikzcd}$$

- Modern SAT solvers can generate resolution proofs using clauses learned by the solver

[ZM03]

Unsatisfiable cores & proof traces

- CNF formula:

$$\mathcal{F} \quad = \quad (\bar{c}) \wedge (\bar{b}) \wedge (\bar{a} \vee c) \wedge (a \vee b) \wedge (a \vee \bar{d}) \wedge (\bar{a} \vee \bar{d})$$

Level	Dec.	Unit Prop.
0	$\emptyset$	$\begin{tikzcd} \bar{b} \arrow[r] & a \arrow[d] \\ \bar{c} \arrow[r] & \perp \end{tikzcd}$

Implication graph with conflict

Unsatisfiable cores & proof traces

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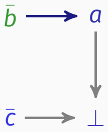
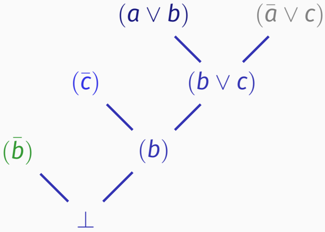
Proof trace $\perp$: $(\bar{a} \vee c)$ $(a \vee b)$ $(\bar{c})$ $(\bar{b})$

Unsatisfiable cores & proof traces

- CNF formula:

$$\mathcal{F} \quad = \quad (\bar{c}) \land (\bar{b}) \land (\bar{a} \lor c) \land (a \lor b) \land (a \lor \bar{d}) \land (\bar{a} \lor \bar{d})$$

Level	Dec.	Unit Prop.
0	$\emptyset$	



Resolution proof follows structure of conflicts

Unsatisfiable cores & proof traces

- CNF formula:

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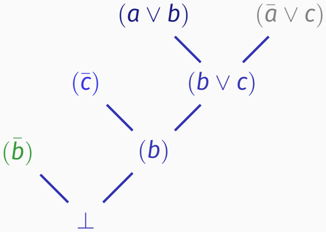
Level	Dec.	Unit Prop.
0	$\emptyset$	$\bar{b} \to a$, $a \to \bot$, $\bar{c} \to \bot$

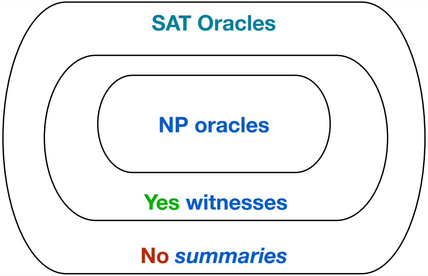
$$\begin{tikzcd} \bar{b} \arrow[r] & a \arrow[d] \\ \bar{c} \arrow[r] & \bot \end{tikzcd}$$



Unsatisfiable subformula (core): $(\bar{c}), (\bar{b}), (\bar{a} \vee c), (a \vee b)$

1 Problem Solving with SAT Oracles

So what are SAT oracles?



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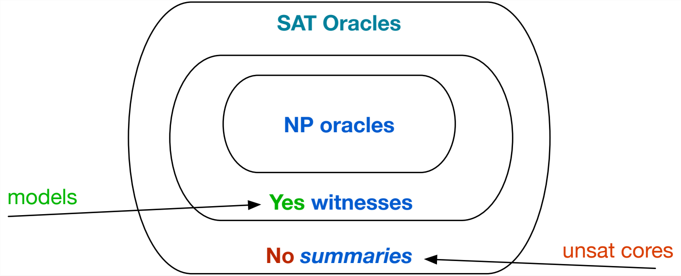

Computing a model

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  FSAT: Compute a model of a satisfiable CNF formula $\mathcal{F}$, using an NP oracle

  - A possible algorithm:
    1. Analyze each variable $x_i \in \{x_1, \ldots, x_n\} = \text{var}(\mathcal{F})$, in order
    2. $i \leftarrow 1$ and $\mathcal{F}_i \triangleq \mathcal{F}$
    3. Call NP oracle on $\mathcal{F}_i \wedge (x_i)$
    4. If answer is **yes**, then $\mathcal{F}_{i+1} \leftarrow \mathcal{F}_i \cup (x_i)$
    5. If answer is **no**, then $\mathcal{F}_{i+1} \leftarrow \mathcal{F}_i \cup (\neg x_i)$
    6. $i \leftarrow i + 1$
    7. If $i \leq n$, then repeat from 3.

Computing a model

- **Q:** How to solve the FSAT problem?

  FSAT: Compute a model of a satisfiable CNF formula $\mathcal{F}$, using an NP oracle

  - A possible algorithm:
    1. Analyze each variable $x_i \in \{x_1, \ldots, x_n\} = \text{var}(\mathcal{F})$, in order
    2. $i \leftarrow 1$ and $\mathcal{F}_i \triangleq \mathcal{F}$
    3. Call NP oracle on $\mathcal{F}_i \wedge (x_i)$
    4. If answer is **yes**, then $\mathcal{F}_{i+1} \leftarrow \mathcal{F}_i \cup (x_i)$
    5. If answer is **no**, then $\mathcal{F}_{i+1} \leftarrow \mathcal{F}_i \cup (\neg x_i)$
    6. $i \leftarrow i + 1$
    7. If $i \leq n$, then repeat from 3.
  - Algorithm needs $|\text{var}(\mathcal{F})|$ calls to an NP oracle

Computing a model

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  - **Note:** Cannot solve FSAT with logarithmic number of NP oracle calls, unless $\mathrm{P} = \mathrm{NP}$ [GF93]

- FSAT is an example of a function problem

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- FSAT is an example of a function problem
  - **Note:** FSAT can be solved with **one** SAT oracle call

Beyond decision problems

Answer	Problem Type

Beyond decision problems

Answer	Problem Type
Yes/No	Decision Problems

Beyond decision problems

Answer	Problem Type
Yes/No	Decision Problems
Some solution	

Beyond decision problems

Answer	Problem Type
Yes/No	Decision Problems
Some solution	Function Problems

Beyond decision problems

Answer	Problem Type
Yes/No	Decision Problems
Some solution	Function Problems
All solutions	

Beyond decision problems

Answer	Problem Type
Yes/No	Decision Problems
Some solution	Function Problems
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Beyond decision problems

Answer	Problem Type
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# solutions	

Beyond decision problems

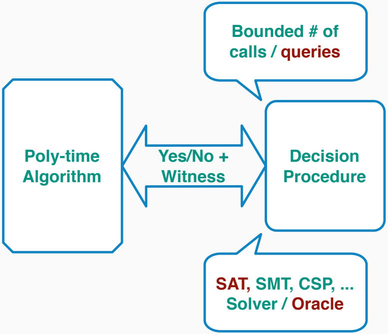
Answer	Problem Type
Yes/No	Decision Problems
Some solution	Function Problems
All solutions	Enumeration Problems
# solutions	Counting Problems

… and beyond NP – decision and function problems

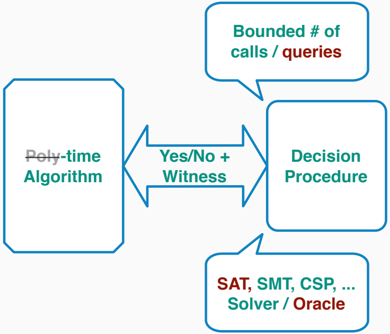
$$
\begin{tikzcd}
& \vdots & \\
\Sigma_3^{\mathsf{p}} & & \Pi_3^{\mathsf{p}} \\
& \Delta_3^{\mathsf{p}} \arrow[ul] \arrow[ur] & \\
\Sigma_2^{\mathsf{p}} \arrow[uu] \arrow[ur] & & \Pi_2^{\mathsf{p}} \arrow[uu] \arrow[ul] \\
& \mathsf{P}^{\mathsf{NP}} = \Delta_2^{\mathsf{p}} \arrow[ul] \arrow[ur] & \\
\mathsf{NP} = \Sigma_1^{\mathsf{p}} \arrow[uu] \arrow[ur] & & \Pi_1^{\mathsf{p}} = \mathsf{coNP} \arrow[uu] \arrow[ul] \\
& \Delta_0^{\mathsf{p}} = \Sigma_0^{\mathsf{p}} = \mathsf{P} = \Pi_0^{\mathsf{p}} = \Delta_1^{\mathsf{p}} \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& \vdots & \\
\mathsf{F}\Sigma_3^{\mathsf{p}} & & \mathsf{F}\Pi_3^{\mathsf{p}} \\
& \mathsf{F}\Delta_3^{\mathsf{p}} \arrow[ul] \arrow[ur] & \\
\mathsf{F}\Sigma_2^{\mathsf{p}} \arrow[uu] \arrow[ur] & & \mathsf{F}\Pi_2^{\mathsf{p}} \arrow[uu] \arrow[ul] \\
& \mathsf{FP}^{\mathsf{NP}} = \mathsf{F}\Delta_2^{\mathsf{p}} \arrow[ul] \arrow[ur] & \\
\mathsf{FNP} = \mathsf{F}\Sigma_1^{\mathsf{p}} \arrow[uu] \arrow[ur] & & \mathsf{F}\Pi_1^{\mathsf{p}} = \mathsf{coFNP} \arrow[uu] \arrow[ul] \\
& \mathsf{F}\Delta_0^{\mathsf{p}} = \mathsf{F}\Sigma_0^{\mathsf{p}} = \mathsf{FP} = \mathsf{F}\Pi_0^{\mathsf{p}} = \mathsf{F}\Delta_1^{\mathsf{p}} \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

Oracle-based problem solving – simple scenario



Oracle-based problem solving – general setting



Many problems to solve – within $\mathrm{FP}^{\mathrm{NP}}$

Answer	Problem Type
Yes/No	Decision Problems
Some solution	Function Problems
All solutions	Enumeration Problems

Many problems to solve – within $\mathrm{FP}^{\mathrm{NP}}$

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Function Problems on Propositional Formulas

MaxSAT **PBO** **...** **WBO** **MinSAT**

Minimal Models **Prime Implicants** **Maximal Models** **Autarkies** **Backbones** **Prime Implicates** **...** **MUSes** **MCSes** **MESes** **Indep. Vars** **MDSes** **Implicant Ext.** **MFSes** **MSSes** **MNSes** **Implicate Ext.** **MCFSes**

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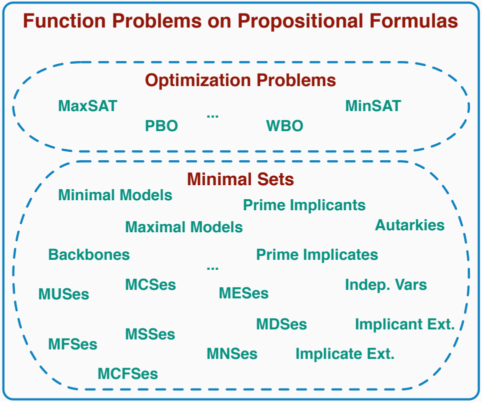
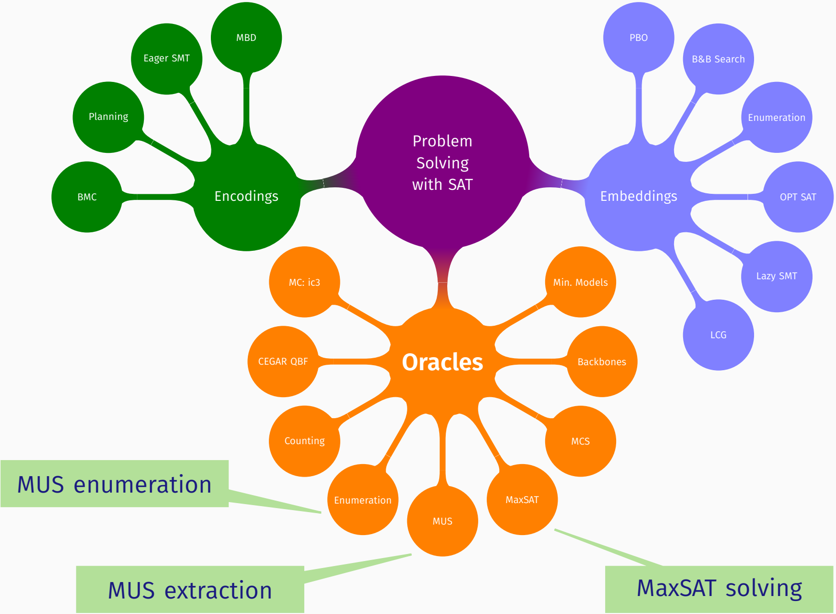


Selection of topics



Outline

Minimal Unsatisfiability

MUS Enumeration

Maximum Satisfiability

Analyzing inconsistency – timetabling

Subject	Day	Time	Room
Intro Prog	Mon	9:00-10:00	6.2.46
Intro AI	Tue	10:00-11:00	8.2.37
Databases	Tue	11:00-12:00	8.2.37
... (hundreds of consistent constraints)			
Linear Alg	Mon	9:00-10:00	6.2.46
Calculus	Tue	10:00-11:00	8.2.37
Adv Calculus	Mon	9:00-10:00	8.2.06
... (hundreds of consistent constraints)			

- Set of constraints consistent / satisfiable?

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Minimality matters!

Unsatisfiable formulas – MUSes & MCSes

- Given $\mathcal{F}$ $(\vDash \perp)$, $\mathcal{M} \subseteq \mathcal{F}$ is a Minimal Unsatisfiable Subset (MUS) iff $\mathcal{M} \vDash \perp$ and $\forall_{\mathcal{M}' \subsetneq \mathcal{M}}, \mathcal{M}' \nvDash \perp$

$$(\neg x_1 \vee \neg x_2) \wedge (x_1) \wedge (x_2) \wedge (\neg x_3 \vee \neg x_4) \wedge (x_3) \wedge (x_4) \wedge (x_5 \vee x_6)$$

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- Given $\mathcal{F}$ $(\vDash \perp)$, $\mathcal{C} \subseteq \mathcal{F}$ is a Minimal Correction Subset (MCS) iff $\mathcal{F} \setminus \mathcal{C} \nvDash \perp$ and $\forall_{\mathcal{C}' \subsetneq \mathcal{C}}, \mathcal{F} \setminus \mathcal{C}' \vDash \perp$. $\mathcal{S} = \mathcal{F} \setminus \mathcal{C}$ is MSS

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- MUSes and minimal hitting sets of MCSes and vice-versa [Rei87, BS05]
  - Easy to see **why**

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  - Easy to see **why**
- How to compute MUSes & MCSes **efficiently** with SAT oracles?

Why it matters?

- Analysis of over-constrained systems
  - Model-based diagnosis [Rei87]
    - Software fault localization
    - Spreadsheet debugging
    - Debugging relational specifications (e.g. Alloy)
    - Type error debugging
    - Axiom pinpointing in description logics
    - ...
  - Model checking of software & hardware systems
  - Inconsistency measurement
  - Minimal models; MinCost SAT; ...
  - ...
- Find minimal relaxations to recover consistency
  - But also minimum relaxations to recover consistency, eg. **MaxSAT**
- **Find minimal explanations of inconsistency**
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Enumeration required!

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  - But also minimum relaxations to recover consistency, eg. **MaxSAT**
- **Find minimal explanations of inconsistency**
  - But also minimum explanations of inconsistency, eg. Smallest MUS

Deletion-based algorithm

Input : Set $\mathcal{F}$
Output: Minimal subset $\mathcal{M}$
begin
  $\mathcal{M} \leftarrow \mathcal{F}$
  foreach $c \in \mathcal{M}$ **do**
    if $\neg\mathrm{SAT}(\mathcal{M} \setminus \{c\})$ **then**
      $\mathcal{M} \leftarrow \mathcal{M} \setminus \{c\}$ // If $\neg\mathrm{SAT}(\mathcal{M} \setminus \{c\})$, then $c \notin$ MUS
  return $\mathcal{M}$ // Final $\mathcal{M}$ is MUS
end

- Number of oracles calls: $\mathcal{O}(m)$ [CD91, BDTW93]

Deletion-based algorithm

Input : Set $\mathcal{F}$
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  foreach $c \in \mathcal{M}$ **do**
    if $\neg\mathrm{SAT}(\mathcal{M} \setminus \{c\})$ **then**
      $\mathcal{M} \leftarrow \mathcal{M} \setminus \{c\}$ // Remove c from $\mathcal{M}$
  return $\mathcal{M}$ // Final $\mathcal{M}$ is MUS
end

Monotonicity implicit & essential!

- Number of oracles calls: $\mathcal{O}(m)$ [CD91, BDTW93]

Deletion – MUS example

c_1	c_2	c_3	c_4	c_5	c_6	c_7
$(\neg x_1 \vee \neg x_2)$	(x_1)	(x_2)	$(\neg x_3 \vee \neg x_4)$	(x_3)	(x_4)	$(x_5 \vee x_6)$

$\mathcal{M}$	$\mathcal{M} \setminus \{c\}$	$\neg\mathrm{SAT}(\mathcal{M} \setminus \{c\})$	Outcome

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$\mathcal{M}$	$\mathcal{M} \setminus \{c\}$	$\neg\mathrm{SAT}(\mathcal{M} \setminus \{c\})$	Outcome
$c_1..c_7$	$c_2..c_7$	1	Drop c_1

Deletion – MUS example

c_1	c_2	c_3	c_4	c_5	c_6	c_7
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$c_4..c_7$	$c_5..c_7$	0	Keep c_4

Deletion – MUS example

c_1	c_2	c_3	c_4	c_5	c_6	c_7
$(\neg x_1 \vee \neg x_2)$	(x_1)	(x_2)	$(\neg x_3 \vee \neg x_4)$	(x_3)	(x_4)	$(x_5 \vee x_6)$

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$c_4..c_7$	$c_5..c_7$	0	Keep c_4
$c_4..c_7$	$c_4 c_6 c_7$	0	Keep c_5

Deletion – MUS example

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$c_4..c_7$	$c_5..c_7$	0	Keep c_4
$c_4..c_7$	$c_4 c_6 c_7$	0	Keep c_5
$c_4..c_7$	$c_4 c_5 c_7$	0	Keep c_6

Deletion – MUS example

c_1	c_2	c_3	c_4	c_5	c_6	c_7
$(\neg x_1 \vee \neg x_2)$	(x_1)	(x_2)	$(\neg x_3 \vee \neg x_4)$	(x_3)	(x_4)	$(x_5 \vee x_6)$

$\mathcal{M}$	$\mathcal{M} \setminus \{c\}$	$\neg\mathrm{SAT}(\mathcal{M} \setminus \{c\})$	Outcome
$c_1..c_7$	$c_2..c_7$	1	Drop c_1
$c_2..c_7$	$c_3..c_7$	1	Drop c_2
$c_3..c_7$	$c_4..c_7$	1	Drop c_3
$c_4..c_7$	$c_5..c_7$	0	Keep c_4
$c_4..c_7$	$c_4c_6c_7$	0	Keep c_5
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$c_4..c_7$	$c_4..c_6$	1	Drop c_7

- MUS: $\{c_4, c_5, c_6\}$

Many MUS algorithms

- Formula $\mathcal{F}$ with m clauses k the size of largest minimal subset

Algorithm	Oracle Calls	Reference
Insertion-based	$\mathcal{O}(k\,m)$	[dSNP88, vMW08]
MCS_MUS	$\mathcal{O}(k\,m)$	[BK15]
Deletion-based	$\mathcal{O}(m)$	[CD91, BDTW93]
Linear insertion	$\mathcal{O}(m)$	[MSL11, BLM12]
Dichotomic	$\mathcal{O}(k\ \log(m))$	[HLSB06]
QuickXplain	$\mathcal{O}(k + k\ \log(\frac{m}{k}))$	[Jun04]
Progression	$\mathcal{O}(k\ \log(1 + \frac{m}{k}))$	[MJB13]

- **Note:** Lower bound in $\mathrm{FP}_{\|}^{\mathrm{NP}}$ and upper bound in $\mathrm{FP}^{\mathrm{NP}}$ [CT95]
- Oracle calls correspond to testing unsatisfiability with SAT solver
- Practical optimizations: clause set trimming; clause set refinement; redundancy removal; (recursive) model rotation

Outline

Minimal Unsatisfiability

MUS Enumeration

Maximum Satisfiability

How to enumerate MUSes?

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1. Standard solution:
   Exploit HS duality between MCSes and MUSes [Rei87, LS08]

   MCSes are MHSes of MUSes and vice-versa

   - Enumerate **all** MCSes and then enumerate **all** MHSes of the MCSes, i.e. compute all the MUSes
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   - And, often we want to enumerate the MUSes

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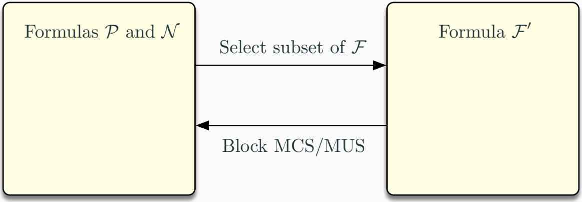
2. Exploit recent advances in 2QBF solving

3. Implicit hitting set dualization [LPMM16]
   - Most effective if MUSes provided to user on-demand

How to enumerate MUSes, preferably?



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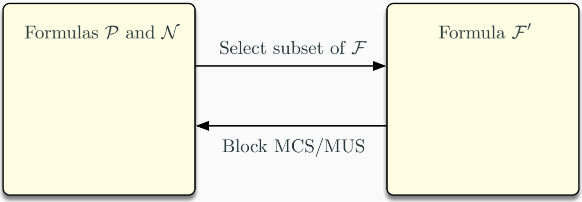

1. Keep sets representing computed MUSes (set $\mathcal{N}$) and MCSes (set $\mathcal{P}$)
2. Compute minimal hitting set (MHS) H of $\mathcal{N}$, subject to $\mathcal{P}$
   - Must **not** repeat MUSes
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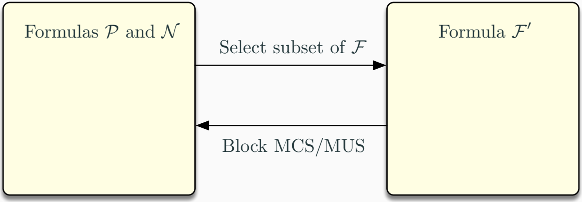

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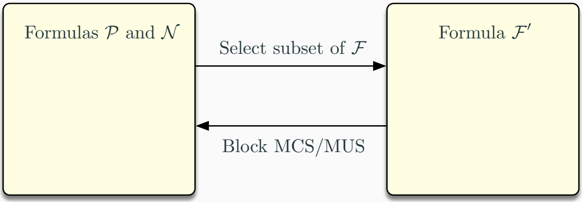

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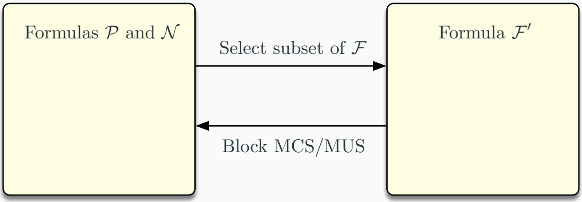

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5. Repeat loop

MARCO/eMUS algorithm

Input: CNF formula $\mathcal{F}$

begin

  $I \leftarrow \{p_i \mid c_i \in \mathcal{F}\}$

  $(\mathcal{P}, \mathcal{N}) \leftarrow (\emptyset, \emptyset)$

  while true **do**

    $(st, H) \leftarrow \text{MinHittingSet}(\mathcal{N}, \mathcal{P})$

    if not st **then return**

    $\mathcal{F}' \leftarrow \{c_i \mid p_i \in I \land p_i \notin H\}$

    if not $\text{SAT}(\mathcal{F}')$ **then**

      $\mathcal{M} \leftarrow \text{ComputeMUS}(\mathcal{F}')$

      $\text{ReportMUS}\ (\mathcal{M})$

      $\mathcal{N} \leftarrow \mathcal{N} \cup \{\neg p_i \mid c_i \in \mathcal{M}\}$

    else

      $\mathcal{P} \leftarrow \mathcal{P} \cup \{p_i \mid p_i \in H\}$

end

An example

MinHS ($\mathcal{N}$)	$\mathcal{F}'$	MUS/MCS
$p_1p_2p_3p_4p_5p_6p_7$	S/U	
1111111	U	$\neg p_1 \vee \neg p_2 \vee \neg p_3$
0111111	U	$\neg p_6 \vee \neg p_7$
0111101	S	$p_1 \vee p_6$
1011101	U	$\neg p_1 \vee \neg p_4 \vee \neg p_5$
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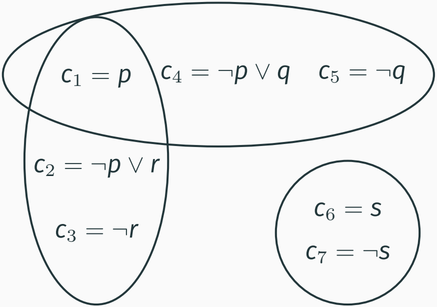

An example

MinHS ($\mathcal{N}$) $p_1p_2p_3p_4p_5p_6p_7$	$\mathcal{F}'$ S/U	MUS/MCS
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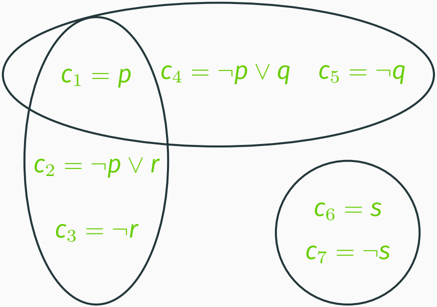

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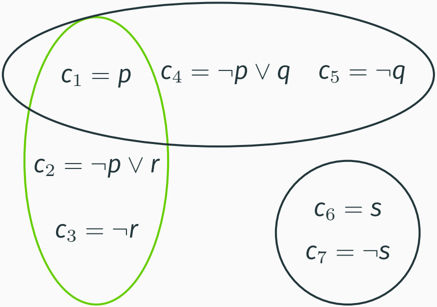

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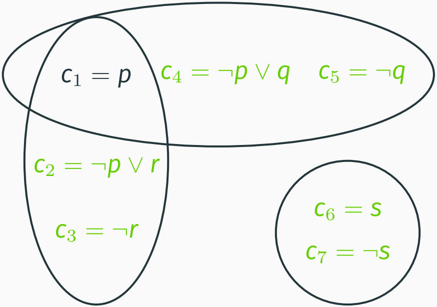

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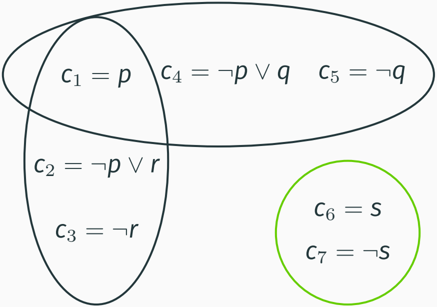

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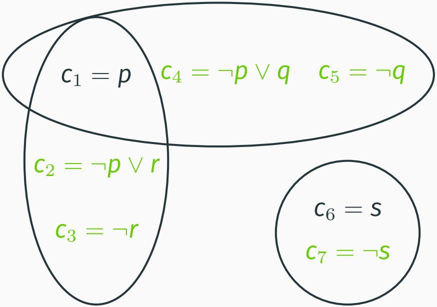

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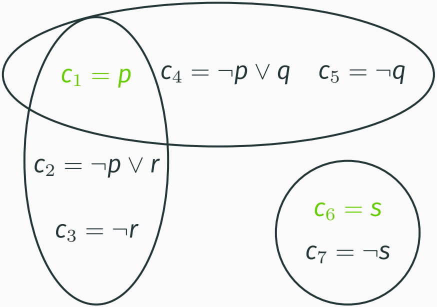

Outline

Minimal Unsatisfiability

MUS Enumeration

Maximum Satisfiability

Recap MaxSAT

$x_6 \vee x_2$	$\neg x_6 \vee x_2$	$\neg x_2 \vee x_1$	$\neg x_1$
$\neg x_6 \vee x_8$	$x_6 \vee \neg x_8$	$x_2 \vee x_4$	$\neg x_4 \vee x_5$
$x_7 \vee x_5$	$\neg x_7 \vee x_5$	$\neg x_5 \vee x_3$	$\neg x_3$

- Given unsatisfiable formula, find largest subset of clauses that is satisfiable

Recap MaxSAT

$\boxed{x_6 \vee x_2}$	$\neg x_6 \vee x_2$	$\neg x_2 \vee x_1$	$\neg x_1$
$\neg x_6 \vee x_8$	$x_6 \vee \neg x_8$	$x_2 \vee x_4$	$\boxed{\neg x_4 \vee x_5}$
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- Given unsatisfiable formula, find largest subset of clauses that is satisfiable
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    - **Note**: Clauses can have weights & there can be hard clauses
- **Many** practical applications [SZGN17]

MaxSAT problem(s)

		Hard Clauses?	
		No	Yes
Weights?	No		
	Yes		

MaxSAT problem(s)

		Hard Clauses?	
		No	Yes
Weights?	No	Plain	Partial
	Yes	Weighted	Weighted Partial

MaxSAT problem(s)

		Hard Clauses? No	Hard Clauses? Yes
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- Must satisfy hard clauses, if any
- Compute set of satisfied soft clauses with maximum cost
  - Without weights, cost of each falsified soft clause is 1
- **Or**, compute set of falsified soft clauses with minimum cost (s.t. hard & remaining soft clauses are satisfied)

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- **Note**: goal is to compute **set** of satisfied (or falsified) clauses; **not** just the cost **!**

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  - After unit propagation:

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$$\{(x), (\neg x \lor y_1), (\neg x \lor y_2), (\neg y_1 \lor \neg z), (\neg y_2 \lor \neg z), (z)\}$$

  - After unit propagation:

$$\{(x), (\neg x \lor y_1), (\neg x \lor y_2), (\neg y_1 \lor \neg z), (\neg y_2 \lor \neg z), (z)\}$$

  - Is 2 the MaxSAT solution??

Issues with MaxSAT

- **Unit propagation is unsound for MaxSAT**
  - Formula with all clauses soft:

$$\{(x), (\neg x \vee y_1), (\neg x \vee y_2), (\neg y_1 \vee \neg z), (\neg y_2 \vee \neg z), (z)\}$$

  - After unit propagation:

$$\{(x), (\neg x \vee y_1), (\neg x \vee y_2), (\neg y_1 \vee \neg z), (\neg y_2 \vee \neg z), (z)\}$$

  - Is 2 the MaxSAT solution??
  - **No!** Enough to either falsify (x) or (z)

Issues with MaxSAT

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  - After unit propagation:

$$\{(x), (\neg x \vee y_1), (\neg x \vee y_2), (\neg y_1 \vee \neg z), (\neg y_2 \vee \neg z), (z)\}$$

  - Is 2 the MaxSAT solution??
  - **No!** Enough to either falsify (x) or (z)
- Cannot use unit propagation

Issues with MaxSAT

- **Unit propagation is unsound for MaxSAT**
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$$\{(x), (\neg x \vee y_1), (\neg x \vee y_2), (\neg y_1 \vee \neg z), (\neg y_2 \vee \neg z), (z)\}$$

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$$\{(x), (\neg x \vee y_1), (\neg x \vee y_2), (\neg y_1 \vee \neg z), (\neg y_2 \vee \neg z), (z)\}$$

  - Is 2 the MaxSAT solution??
  - **No!** Enough to either falsify (x) or (z)
- Cannot use unit propagation
- Cannot learn clauses (using unit propagation)

Issues with MaxSAT

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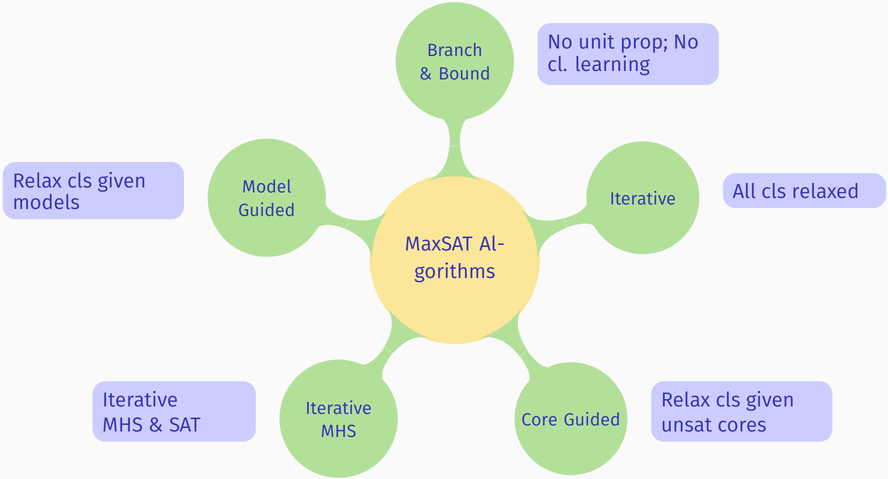
$$\{(x), (\neg x \vee y_1), (\neg x \vee y_2), (\neg y_1 \vee \neg z), (\neg y_2 \vee \neg z), (z)\}$$

  - After unit propagation:

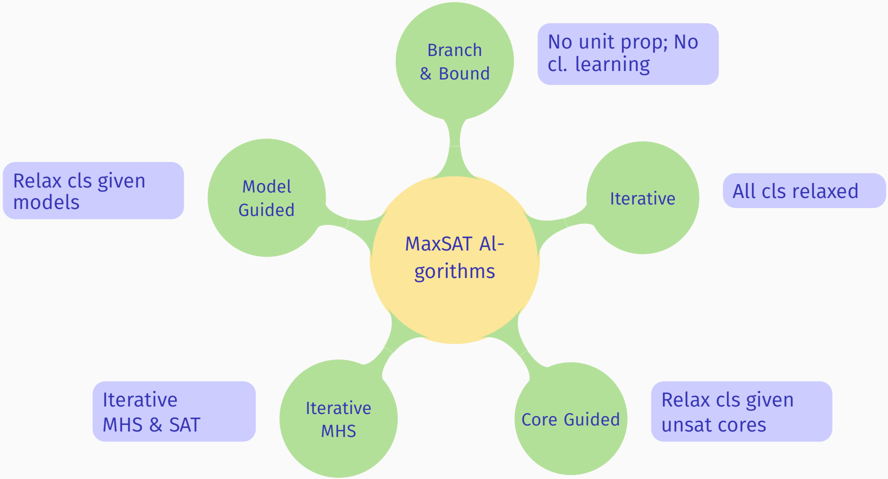
$$\{(x), (\neg x \vee y_1), (\neg x \vee y_2), (\neg y_1 \vee \neg z), (\neg y_2 \vee \neg z), (z)\}$$

  - Is 2 the MaxSAT solution??
  - **No!** Enough to either falsify (x) or (z)
- Cannot use unit propagation
- Cannot learn clauses (using unit propagation)
- Need to solve MaxSAT using different techniques

Many MaxSAT approaches

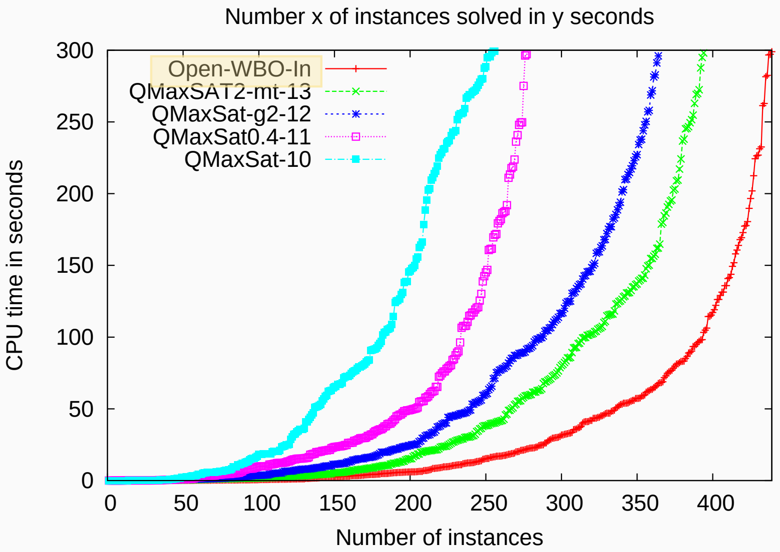


Many MaxSAT approaches



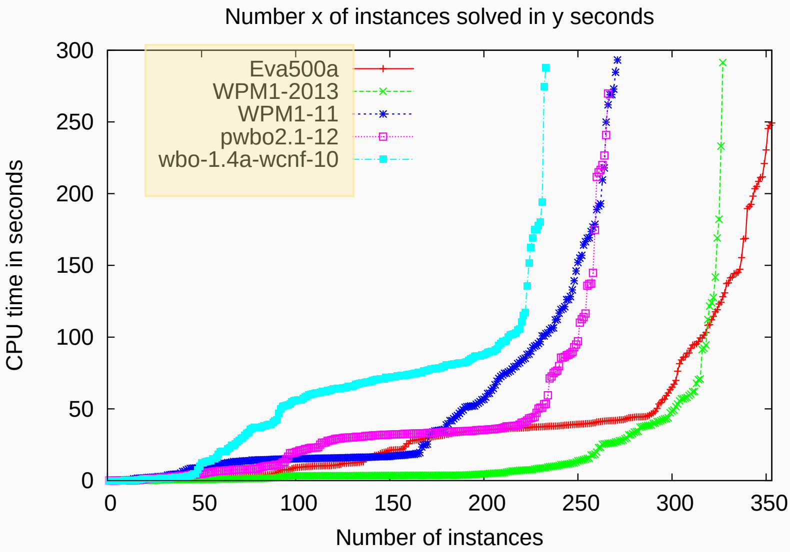
- For practical (**industrial**) instances: **core-guided** & **iterative MHS** approaches are the most effective [MaxSAT14]

Core-guided solver performance – partial



Source: [MaxSAT 2014 organizers]

Core-guided solver performance – weighted partial



Source: [MaxSAT 2014 organizers]

Outline

Minimal Unsatisfiability

MUS Enumeration

Maximum Satisfiability

- Iterative SAT Solving
- Core-Guided Algorithms
- Minimum Hitting Sets

Basic MaxSAT with iterative SAT solving

$x_6 \lor x_2$	$\neg x_6 \lor x_2$	$\neg x_2 \lor x_1$	$\neg x_1$
$\neg x_6 \lor x_8$	$x_6 \lor \neg x_8$	$x_2 \lor x_4$	$\neg x_4 \lor x_5$
$x_7 \lor x_5$	$\neg x_7 \lor x_5$	$\neg x_5 \lor x_3$	$\neg x_3$

Example CNF formula

Basic MaxSAT with iterative SAT solving

$x_6 \vee x_2 \vee r_1$	$\neg x_6 \vee x_2 \vee r_2$	$\neg x_2 \vee x_1 \vee r_3$	$\neg x_1 \vee r_4$
$\neg x_6 \vee x_8 \vee r_5$	$x_6 \vee \neg x_8 \vee r_6$	$x_2 \vee x_4 \vee r_7$	$\neg x_4 \vee x_5 \vee r_8$
$x_7 \vee x_5 \vee r_9$	$\neg x_7 \vee x_5 \vee r_{10}$	$\neg x_5 \vee x_3 \vee r_{11}$	$\neg x_3 \vee r_{12}$

$\sum_{i=1}^{12} r_i \leq 12$

Relax all clauses; Set $UB = 12 + 1$

Basic MaxSAT with iterative SAT solving

$x_6 \vee x_2 \vee r_1$	$\neg x_6 \vee x_2 \vee r_2$	$\neg x_2 \vee x_1 \vee r_3$	$\neg x_1 \vee r_4$
$\neg x_6 \vee x_8 \vee r_5$	$x_6 \vee \neg x_8 \vee r_6$	$x_2 \vee x_4 \vee r_7$	$\neg x_4 \vee x_5 \vee r_8$
$x_7 \vee x_5 \vee r_9$	$\neg x_7 \vee x_5 \vee r_{10}$	$\neg x_5 \vee x_3 \vee r_{11}$	$\neg x_3 \vee r_{12}$

$\sum_{i=1}^{12} r_i \leq 12$

Formula is SAT; E.g. all $x_i = 0$ and $r_1 = r_7 = r_9 = 1$ (i.e. cost $= 3$)

Basic MaxSAT with iterative SAT solving

$x_6 \vee x_2 \vee r_1$	$\neg x_6 \vee x_2 \vee r_2$	$\neg x_2 \vee x_1 \vee r_3$	$\neg x_1 \vee r_4$
$\neg x_6 \vee x_8 \vee r_5$	$x_6 \vee \neg x_8 \vee r_6$	$x_2 \vee x_4 \vee r_7$	$\neg x_4 \vee x_5 \vee r_8$
$x_7 \vee x_5 \vee r_9$	$\neg x_7 \vee x_5 \vee r_{10}$	$\neg x_5 \vee x_3 \vee r_{11}$	$\neg x_3 \vee r_{12}$

$$\sum_{i=1}^{12} r_i \leq 2$$

Refine $UB = 3$

Basic MaxSAT with iterative SAT solving

$x_6 \vee x_2 \vee r_1$	$\neg x_6 \vee x_2 \vee r_2$	$\neg x_2 \vee x_1 \vee r_3$	$\neg x_1 \vee r_4$
$\neg x_6 \vee x_8 \vee r_5$	$x_6 \vee \neg x_8 \vee r_6$	$x_2 \vee x_4 \vee r_7$	$\neg x_4 \vee x_5 \vee r_8$
$x_7 \vee x_5 \vee r_9$	$\neg x_7 \vee x_5 \vee r_{10}$	$\neg x_5 \vee x_3 \vee r_{11}$	$\neg x_3 \vee r_{12}$

$\sum_{i=1}^{12} r_i \leq 2$

Formula is SAT; E.g. $x_1 = x_2 = 1$; $x_3 = \ldots = x_8 = 0$ and $r_4 = r_9 = 1$ (i.e. cost $= 2$)

Basic MaxSAT with iterative SAT solving

$x_6 \vee x_2 \vee r_1$	$\neg x_6 \vee x_2 \vee r_2$	$\neg x_2 \vee x_1 \vee r_3$	$\neg x_1 \vee r_4$
$\neg x_6 \vee x_8 \vee r_5$	$x_6 \vee \neg x_8 \vee r_6$	$x_2 \vee x_4 \vee r_7$	$\neg x_4 \vee x_5 \vee r_8$
$x_7 \vee x_5 \vee r_9$	$\neg x_7 \vee x_5 \vee r_{10}$	$\neg x_5 \vee x_3 \vee r_{11}$	$\neg x_3 \vee r_{12}$

$\sum_{i=1}^{12} r_i \leq 1$

Refine $UB = 2$

Basic MaxSAT with iterative SAT solving

$x_6 \vee x_2 \vee r_1$	$\neg x_6 \vee x_2 \vee r_2$	$\neg x_2 \vee x_1 \vee r_3$	$\neg x_1 \vee r_4$
$\neg x_6 \vee x_8 \vee r_5$	$x_6 \vee \neg x_8 \vee r_6$	$x_2 \vee x_4 \vee r_7$	$\neg x_4 \vee x_5 \vee r_8$
$x_7 \vee x_5 \vee r_9$	$\neg x_7 \vee x_5 \vee r_{10}$	$\neg x_5 \vee x_3 \vee r_{11}$	$\neg x_3 \vee r_{12}$

$\sum_{i=1}^{12} r_i \leq 1$

Formula is UNSAT; terminate

Basic MaxSAT with iterative SAT solving

$x_6 \vee x_2 \vee r_1$	$\neg x_6 \vee x_2 \vee r_2$	$\neg x_2 \vee x_1 \vee r_3$	$\neg x_1 \vee r_4$
$\neg x_6 \vee x_8 \vee r_5$	$x_6 \vee \neg x_8 \vee r_6$	$x_2 \vee x_4 \vee r_7$	$\neg x_4 \vee x_5 \vee r_8$
$x_7 \vee x_5 \vee r_9$	$\neg x_7 \vee x_5 \vee r_{10}$	$\neg x_5 \vee x_3 \vee r_{11}$	$\neg x_3 \vee r_{12}$

$\sum_{i=1}^{12} r_i \leq 1$

MaxSAT solution is last satisfied UB: $UB = 2$

Basic MaxSAT with iterative SAT solving

$x_6 \vee x_2 \vee r_1$ $\quad \neg x_6 \vee x_2 \vee r_2$ $\quad \neg x_2 \vee x_1 \vee r_3$ $\quad \neg x_1 \vee r_4$

$\neg x_6 \vee x_8 \vee r_5$ $\quad x_6 \vee \neg x_8 \vee r_6$ $\quad x_2 \vee x_4 \vee r_7$ $\quad \neg x_4 \vee x_5 \vee r_8$

$x_7 \vee x_5 \vee r_9$ $\quad \neg x_7 \vee x_5 \vee r_{10}$ $\quad \neg x_5 \vee x_3 \vee r_{11}$ $\quad \neg x_3 \vee r_{12}$

$\sum_{i=1}^{12} r_i \leq 1$

MaxSAT solution is last satisfied UB: $UB = 2$

AtMost$k$/PB constraints over **all** relaxation variables

All (possibly many) soft clauses relaxed

Outline

MSU3 core-guided algorithm

$x_6 \vee x_2$	$\neg x_6 \vee x_2$	$\neg x_2 \vee x_1$	$\neg x_1$
$\neg x_6 \vee x_8$	$x_6 \vee \neg x_8$	$x_2 \vee x_4$	$\neg x_4 \vee x_5$
$x_7 \vee x_5$	$\neg x_7 \vee x_5$	$\neg x_5 \vee x_3$	$\neg x_3$

Example CNF formula

MSU3 core-guided algorithm

$x_6 \vee x_2$	$\neg x_6 \vee x_2$	$\neg x_2 \vee x_1$	$\neg x_1$
$\neg x_6 \vee x_8$	$x_6 \vee \neg x_8$	$x_2 \vee x_4$	$\neg x_4 \vee x_5$
$x_7 \vee x_5$	$\neg x_7 \vee x_5$	$\neg x_5 \vee x_3$	$\neg x_3$

Formula is UNSAT; OPT $\leq |\varphi| - 1$; Get unsat core

MSU3 core-guided algorithm

$x_6 \vee x_2$ $\neg x_6 \vee x_2$ $\neg x_2 \vee x_1 \vee r_1$ $\neg x_1 \vee r_2$

$\neg x_6 \vee x_8$ $x_6 \vee \neg x_8$ $x_2 \vee x_4 \vee r_3$ $\neg x_4 \vee x_5 \vee r_4$

$x_7 \vee x_5$ $\neg x_7 \vee x_5$ $\neg x_5 \vee x_3 \vee r_5$ $\neg x_3 \vee r_6$

$\sum_{i=1}^{6} r_i \leq 1$

Add relaxation variables and AtMost$k$, $k = 1$, constraint

MSU3 core-guided algorithm

$x_6 \vee x_2$	$\neg x_6 \vee x_2$	$\neg x_2 \vee x_1 \vee r_1$	$\neg x_1 \vee r_2$
$\neg x_6 \vee x_8$	$x_6 \vee \neg x_8$	$x_2 \vee x_4 \vee r_3$	$\neg x_4 \vee x_5 \vee r_4$
$x_7 \vee x_5$	$\neg x_7 \vee x_5$	$\neg x_5 \vee x_3 \vee r_5$	$\neg x_3 \vee r_6$

$\sum_{i=1}^{6} r_i \leq 1$

Formula is (again) UNSAT; OPT $\leq |\varphi| - 2$; Get unsat core

MSU3 core-guided algorithm

$x_6 \vee x_2 \vee r_7$	$\neg x_6 \vee x_2 \vee r_8$	$\neg x_2 \vee x_1 \vee r_1$	$\neg x_1 \vee r_2$
$\neg x_6 \vee x_8$	$x_6 \vee \neg x_8$	$x_2 \vee x_4 \vee r_3$	$\neg x_4 \vee x_5 \vee r_4$
$x_7 \vee x_5 \vee r_9$	$\neg x_7 \vee x_5 \vee r_{10}$	$\neg x_5 \vee x_3 \vee r_5$	$\neg x_3 \vee r_6$

$\sum_{i=1}^{10} r_i \leq 2$

Add new relaxation variables and update AtMost$k$, k=2, constraint

MSU3 core-guided algorithm

$x_6 \vee x_2 \vee r_7$	$\neg x_6 \vee x_2 \vee r_8$	$\neg x_2 \vee x_1 \vee r_1$	$\neg x_1 \vee r_2$
$\neg x_6 \vee x_8$	$x_6 \vee \neg x_8$	$x_2 \vee x_4 \vee r_3$	$\neg x_4 \vee x_5 \vee r_4$
$x_7 \vee x_5 \vee r_9$	$\neg x_7 \vee x_5 \vee r_{10}$	$\neg x_5 \vee x_3 \vee r_5$	$\neg x_3 \vee r_6$

$\sum_{i=1}^{10} r_i \leq 2$

Instance is now SAT

MSU3 core-guided algorithm

$x_6 \vee x_2 \vee r_7$	$\neg x_6 \vee x_2 \vee r_8$	$\neg x_2 \vee x_1 \vee r_1$	$\neg x_1 \vee r_2$
$\neg x_6 \vee x_8$	$x_6 \vee \neg x_8$	$x_2 \vee x_4 \vee r_3$	$\neg x_4 \vee x_5 \vee r_4$
$x_7 \vee x_5 \vee r_9$	$\neg x_7 \vee x_5 \vee r_{10}$	$\neg x_5 \vee x_3 \vee r_5$	$\neg x_3 \vee r_6$

$\sum_{i=1}^{10} r_i \leq 2$

MaxSAT solution is $|\varphi| - \mathcal{I} = 12 - 2 = 10$

MSU3 core-guided algorithm

$x_6 \vee x_2 \vee r_7$ $\quad \neg x_6 \vee x_2 \vee r_8$ $\quad \neg x_2 \vee x_1 \vee r_1$ $\quad \neg x_1 \vee r_2$

$\neg x_6 \vee x_8$ $\quad x_6 \vee \neg x_8$ $\quad x_2 \vee x_4 \vee r_3$ $\quad \neg x_4 \vee x_5 \vee r_4$

$x_7 \vee x_5 \vee r_9$ $\quad \neg x_7 \vee x_5 \vee r_{10}$ $\quad \neg x_5 \vee x_3 \vee r_5$ $\quad \neg x_3 \vee r_6$

$\sum_{i=1}^{10} r_i \leq 2$

MaxSAT solution is $|\varphi| - \mathcal{I} = 12 - 2 = 10$

AtMost$k$/PB constraints used

Relaxed soft clauses become **hard**

MSU3 core-guided algorithm

$x_6 \vee x_2 \vee r_7$ $\quad \neg x_6 \vee x_2 \vee r_8$ $\quad \neg x_2 \vee x_1 \vee r_1$ $\quad \neg x_1 \vee r_2$

$\boxed{\neg x_6 \vee x_8 \quad x_6 \vee \neg x_8}$ $\quad x_2 \vee x_4 \vee r_3$ $\quad \neg x_4 \vee x_5 \vee r_4$

$x_7 \vee x_5 \vee r_9$ $\quad \neg x_7 \vee x_5 \vee r_{10}$ $\quad \neg x_5 \vee x_3 \vee r_5$ $\quad \neg x_3 \vee r_6$

$\sum_{i=1}^{10} r_i \leq 2$

MaxSAT solution is $|\varphi| - \mathcal{I} = 12 - 2 = 10$

AtMost$k$/PB constraints used

Some clauses not relaxed

Relaxed soft clauses become **hard**

Outline

MHS approach for MaxSAT

$$c_1 = x_6 \vee x_2 \qquad c_2 = \neg x_6 \vee x_2 \qquad c_3 = \neg x_2 \vee x_1 \qquad c_4 = \neg x_1$$

$$c_5 = \neg x_6 \vee x_8 \qquad c_6 = x_6 \vee \neg x_8 \qquad c_7 = x_2 \vee x_4 \qquad c_8 = \neg x_4 \vee x_5$$

$$c_9 = x_7 \vee x_5 \qquad c_{10} = \neg x_7 \vee x_5 \qquad c_{11} = \neg x_5 \vee x_3 \qquad c_{12} = \neg x_3$$

$$\mathcal{K} = \emptyset$$

- Find MHS of $\mathcal{K}$:

MHS approach for MaxSAT

$$c_1 = x_6 \vee x_2 \qquad c_2 = \neg x_6 \vee x_2 \qquad c_3 = \neg x_2 \vee x_1 \qquad c_4 = \neg x_1$$

$$c_5 = \neg x_6 \vee x_8 \qquad c_6 = x_6 \vee \neg x_8 \qquad c_7 = x_2 \vee x_4 \qquad c_8 = \neg x_4 \vee x_5$$

$$c_9 = x_7 \vee x_5 \qquad c_{10} = \neg x_7 \vee x_5 \qquad c_{11} = \neg x_5 \vee x_3 \qquad c_{12} = \neg x_3$$

$$\mathcal{K} = \emptyset$$

- Find MHS of $\mathcal{K}$: $\emptyset$

MHS approach for MaxSAT

$$c_1 = x_6 \vee x_2 \qquad c_2 = \neg x_6 \vee x_2 \qquad c_3 = \neg x_2 \vee x_1 \qquad c_4 = \neg x_1$$

$$c_5 = \neg x_6 \vee x_8 \qquad c_6 = x_6 \vee \neg x_8 \qquad c_7 = x_2 \vee x_4 \qquad c_8 = \neg x_4 \vee x_5$$

$$c_9 = x_7 \vee x_5 \qquad c_{10} = \neg x_7 \vee x_5 \qquad c_{11} = \neg x_5 \vee x_3 \qquad c_{12} = \neg x_3$$

$$\mathcal{K} = \emptyset$$

- Find MHS of $\mathcal{K}$: $\emptyset$
- SAT($\mathcal{F} \setminus \emptyset$)?

MHS approach for MaxSAT

$$c_1 = x_6 \vee x_2 \qquad c_2 = \neg x_6 \vee x_2 \qquad c_3 = \neg x_2 \vee x_1 \qquad c_4 = \neg x_1$$

$$c_5 = \neg x_6 \vee x_8 \qquad c_6 = x_6 \vee \neg x_8 \qquad c_7 = x_2 \vee x_4 \qquad c_8 = \neg x_4 \vee x_5$$

$$c_9 = x_7 \vee x_5 \qquad c_{10} = \neg x_7 \vee x_5 \qquad c_{11} = \neg x_5 \vee x_3 \qquad c_{12} = \neg x_3$$

$$\mathcal{K} = \emptyset$$

- Find MHS of $\mathcal{K}$: $\emptyset$
- SAT($\mathcal{F} \setminus \emptyset$)? No

MHS approach for MaxSAT

$c_1 = x_6 \vee x_2$ $\quad c_2 = \neg x_6 \vee x_2$ $\quad c_3 = \neg x_2 \vee x_1$ $\quad c_4 = \neg x_1$

$c_5 = \neg x_6 \vee x_8$ $\quad c_6 = x_6 \vee \neg x_8$ $\quad c_7 = x_2 \vee x_4$ $\quad c_8 = \neg x_4 \vee x_5$

$c_9 = x_7 \vee x_5$ $\quad c_{10} = \neg x_7 \vee x_5$ $\quad c_{11} = \neg x_5 \vee x_3$ $\quad c_{12} = \neg x_3$

$$\mathcal{K} = \emptyset$$

- Find MHS of $\mathcal{K}$: $\emptyset$
- SAT($\mathcal{F} \setminus \emptyset$)? No
- Core of $\mathcal{F}$: $\{c_1, c_2, c_3, c_4\}$

MHS approach for MaxSAT

$$c_1 = x_6 \lor x_2 \qquad c_2 = \neg x_6 \lor x_2 \qquad c_3 = \neg x_2 \lor x_1 \qquad c_4 = \neg x_1$$

$$c_5 = \neg x_6 \lor x_8 \qquad c_6 = x_6 \lor \neg x_8 \qquad c_7 = x_2 \lor x_4 \qquad c_8 = \neg x_4 \lor x_5$$

$$c_9 = x_7 \lor x_5 \qquad c_{10} = \neg x_7 \lor x_5 \qquad c_{11} = \neg x_5 \lor x_3 \qquad c_{12} = \neg x_3$$

$$\mathcal{K} = \{\{c_1, c_2, c_3, c_4\}\}$$

- Find MHS of $\mathcal{K}$: $\emptyset$
- SAT($\mathcal{F} \setminus \emptyset$)? No
- Core of $\mathcal{F}$: $\{c_1, c_2, c_3, c_4\}$. Update $\mathcal{K}$

MHS approach for MaxSAT

$$c_1 = x_6 \vee x_2 \qquad c_2 = \neg x_6 \vee x_2 \qquad c_3 = \neg x_2 \vee x_1 \qquad c_4 = \neg x_1$$

$$c_5 = \neg x_6 \vee x_8 \qquad c_6 = x_6 \vee \neg x_8 \qquad c_7 = x_2 \vee x_4 \qquad c_8 = \neg x_4 \vee x_5$$

$$c_9 = x_7 \vee x_5 \qquad c_{10} = \neg x_7 \vee x_5 \qquad c_{11} = \neg x_5 \vee x_3 \qquad c_{12} = \neg x_3$$

$$\mathcal{K} = \{\{c_1, c_2, c_3, c_4\}\}$$

- Find MHS of $\mathcal{K}$:

MHS approach for MaxSAT

$$c_1 = x_6 \vee x_2 \qquad c_2 = \neg x_6 \vee x_2 \qquad c_3 = \neg x_2 \vee x_1 \qquad c_4 = \neg x_1$$

$$c_5 = \neg x_6 \vee x_8 \qquad c_6 = x_6 \vee \neg x_8 \qquad c_7 = x_2 \vee x_4 \qquad c_8 = \neg x_4 \vee x_5$$

$$c_9 = x_7 \vee x_5 \qquad c_{10} = \neg x_7 \vee x_5 \qquad c_{11} = \neg x_5 \vee x_3 \qquad c_{12} = \neg x_3$$

$$\mathcal{K} = \{\{c_1, c_2, c_3, c_4\}\}$$

- Find MHS of $\mathcal{K}$: E.g. $\{c_1\}$

MHS approach for MaxSAT

$$c_1 = x_6 \vee x_2 \qquad c_2 = \neg x_6 \vee x_2 \qquad c_3 = \neg x_2 \vee x_1 \qquad c_4 = \neg x_1$$

$$c_5 = \neg x_6 \vee x_8 \qquad c_6 = x_6 \vee \neg x_8 \qquad c_7 = x_2 \vee x_4 \qquad c_8 = \neg x_4 \vee x_5$$

$$c_9 = x_7 \vee x_5 \qquad c_{10} = \neg x_7 \vee x_5 \qquad c_{11} = \neg x_5 \vee x_3 \qquad c_{12} = \neg x_3$$

$$\mathcal{K} = \{\{c_1, c_2, c_3, c_4\}\}$$

- Find MHS of $\mathcal{K}$: E.g. $\{c_1\}$
- SAT($\mathcal{F} \setminus \{c_1\}$)?

MHS approach for MaxSAT

$$c_1 = x_6 \vee x_2 \qquad c_2 = \neg x_6 \vee x_2 \qquad c_3 = \neg x_2 \vee x_1 \qquad c_4 = \neg x_1$$

$$c_5 = \neg x_6 \vee x_8 \qquad c_6 = x_6 \vee \neg x_8 \qquad c_7 = x_2 \vee x_4 \qquad c_8 = \neg x_4 \vee x_5$$

$$c_9 = x_7 \vee x_5 \qquad c_{10} = \neg x_7 \vee x_5 \qquad c_{11} = \neg x_5 \vee x_3 \qquad c_{12} = \neg x_3$$

$$\mathcal{K} = \{\{c_1, c_2, c_3, c_4\}\}$$

- Find MHS of $\mathcal{K}$: E.g. $\{c_1\}$
- SAT($\mathcal{F} \setminus \{c_1\}$)? No

MHS approach for MaxSAT

$c_1 = x_6 \vee x_2$	$c_2 = \neg x_6 \vee x_2$	$c_3 = \neg x_2 \vee x_1$	$c_4 = \neg x_1$
$c_5 = \neg x_6 \vee x_8$	$c_6 = x_6 \vee \neg x_8$	$c_7 = x_2 \vee x_4$	$c_8 = \neg x_4 \vee x_5$
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$$\mathcal{K} = \{\{c_1, c_2, c_3, c_4\}\}$$

- Find MHS of $\mathcal{K}$: E.g. $\{c_1\}$
- SAT($\mathcal{F} \setminus \{c_1\}$)? No
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MHS approach for MaxSAT

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MHS approach for MaxSAT

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- SAT($\mathcal{F} \setminus \{c_4, c_9\}$)?

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- Find MHS of $\mathcal{K}$: E.g. $\{c_4, c_9\}$
- SAT($\mathcal{F} \setminus \{c_4, c_9\}$)? Yes
- Terminate & return 2

MaxSAT solving with SAT oracles – a sample

- A sample of recent algorithms:

Algorithm	# Oracle Queries	Reference
Linear search SU	Exponential***	[BP10]
Binary search	Linear*	[FM06]
FM/WMSU1/WPM1	Exponential**	[FM06, MP08, MMSP09, ABL09, ABGL12]
WPM2	Exponential**	[ABL10, ABL13]
Bin-Core-Dis	Linear	[HMM11, MHM12]
Iterative MHS	Exponential	[DB11, DB13a, DB13b]

* $\mathcal{O}(\log m)$ queries with SAT oracle, for (partial) unweighted MaxSAT

** Weighted case; depends on computed cores

*** On # bits of problem instance (due to weights)

- But also additional recent work:
  - Progression [IMM+14]
  - Soft cardinality constraints (OLL) [MDM14, MIM14]
    - Recent implementation (RC2, using PySAT) won 2018 MaxSAT Evaluation
  - MaxSAT resolution [NB14]
  - ...

2 Exploring With SAT Oracles

Incremental SAT solving

- SAT solver often called multiple times on related formulas
- It helps to make incremental changes **&** remember already learned clauses (that still hold)

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$\mathfrak{c}_i$	$\mathfrak{c}_i \vee \overline{s}_i$

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  - To remove clause: add unit $(\overline{s_i})$
  - **Any** learned clause contains explanation given working assumptions (more next)

An example

$$\mathcal{B} = \{(\bar{a} \vee b), (\bar{a} \vee c)\}$$
$$\mathcal{S} = \{(a \vee \overline{s_1}), (\bar{b} \vee \bar{c} \vee \overline{s_2}), (a \vee \bar{c} \vee \overline{s_3}), (a \vee \bar{b} \vee \overline{s_4})\}$$

- Background knowledge $\mathcal{B}$: final clauses, i.e. no indicator variables
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- Background knowledge $\mathcal{B}$: final clauses, i.e. no indicator variables
- Soft clauses $\mathcal{S}$: add indicator variables $\{s_1, s_2, s_3, s_4\}$
- E.g. given assumptions $\{s_1 = 1, s_2 = 0, s_3 = 0, s_4 = 1\}$, SAT solver handles formula:

$$\mathcal{F} = \{(\bar{a} \vee b), (\bar{a} \vee c), (a), (a \vee \bar{b})\}$$

which is satisfiable

Quiz – what happens in this case?

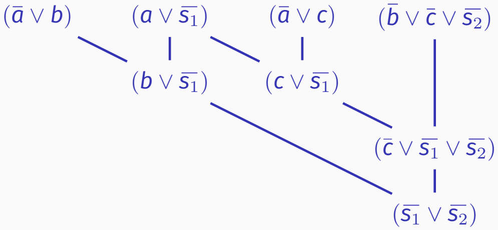
$$\mathcal{B} = \{(\bar{a} \vee b), (\bar{a} \vee c)\}$$
$$\mathcal{S} = \{(a \vee \overline{s_1}), (\bar{b} \vee \bar{c} \vee \overline{s_2}), (a \vee \bar{c} \vee \overline{s_3}), (a \vee \bar{b} \vee \overline{s_4})\}$$

- Given assumptions $\{s_1 = 1, s_2 = 1, s_3 = 1, s_4 = 1\}$?

Quiz – what happens in this case?

$$\mathcal{B} = \{(\bar{a} \lor b), (\bar{a} \lor c)\}$$
$$\mathcal{S} = \{(a \lor \overline{s_1}), (\bar{b} \lor \bar{c} \lor \overline{s_2}), (a \lor \bar{c} \lor \overline{s_3}), (a \lor \bar{b} \lor \overline{s_4})\}$$

- Given assumptions $\{s_1 = 1, s_2 = 1, s_3 = 1, s_4 = 1\}$?



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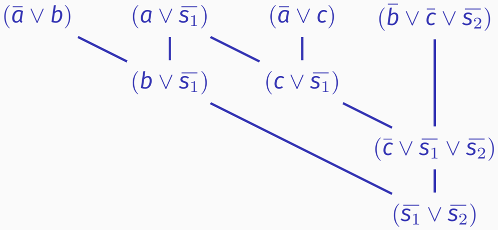
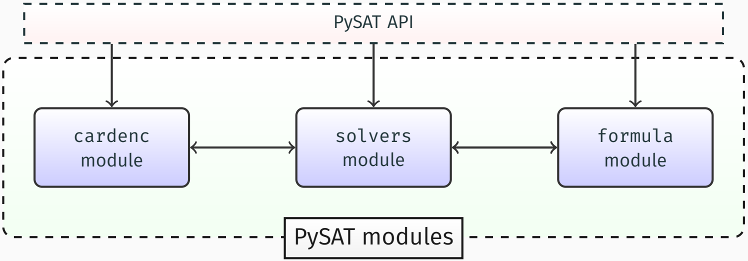


- Unsatisfiable core: 1^{st} and 2^{nd} clauses of $\mathcal{S}$, given $\mathcal{B}$

Overview of PySAT

[IMM18]



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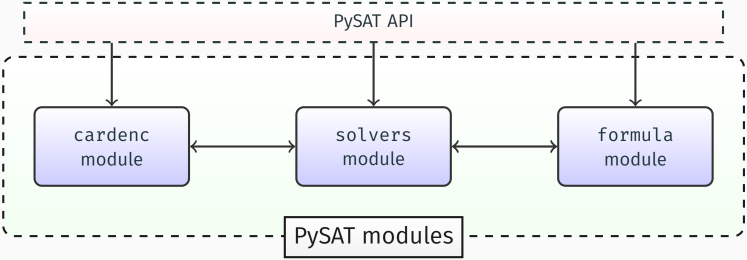

- Open source, available on github

Overview of PySAT

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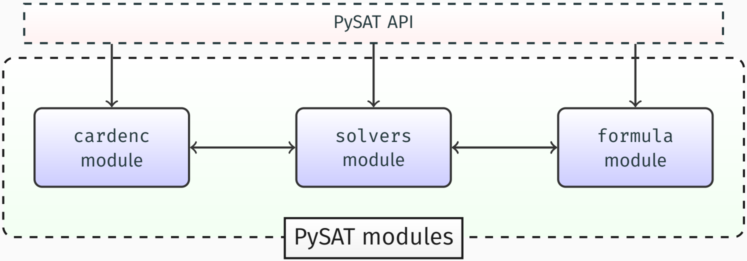

- Open source, available on github
- Comprehensive list of SAT solvers
- Comprehensive list of cardinality encodings
- Fairly comprehensive documentation
- Several use cases

Available solvers

Solver	Version
Glucose	3.0
Glucose	4.1
Lingeling	bbc-9230380-160707
Minicard	1.2
Minisat	2.2 release
Minisat	GitHub version
MapleCM	SAT competition 2018
Maplesat	MapleCOMSPS_LRB
...	...

- Solvers can either be used incrementally or non-incrementally
- Tools can use multiple solvers, e.g. for hitting set dualization or CEGAR-based QBF solving
- **URL**: `https://pysathq.github.io/docs/html/api/solvers.html`

Formula manipulation

Features
CNF & Weighted CNF (WCNF)
Read formulas from file/string
Write formulas to file
Append clauses to formula
Negate CNF formulas
Translate between CNF and WCNF
ID manager

- **URL**: `https://pysathq.github.io/docs/html/api/formula.html`

Available cardinality encodings

Name	Type
pairwise	AtMost1
bitwise	AtMost1
ladder	AtMost1
sequential counter	AtMost$k$
sorting network	AtMost$k$
cardinality network	AtMost$k$
totalizer	AtMost$k$
mtotalizer	AtMost$k$
kmtotalizer	AtMost$k$

- Also AtLeast$K$ and Equals$K$ constraints
- **URL**: `https://pysathq.github.io/docs/html/api/card.html`

Installation & info

- Installation:
  `$ [sudo] pip2|pip3 install python-sat`

- Website: `https://pysathq.github.io/`

Basic interface – Python3 shell

```
>>> from pysat.card import *
>>> am1 = CardEnc.atmost(lits=[1, -2, 3], encoding=EncType.pairwise)
>>> print(am1.clauses)
[[-1, 2], [-1, -3], [2, -3]]
>>>
>>> from pysat.solvers import Solver
>>> with Solver(name='m22', bootstrap_with=am1.clauses) as s:
...     if s.solve(assumptions=[1, 2, 3]) == False:
...         print(s.get_core())
[3, 1]
```

Basic interface – Python3 script

```
#!/usr/local/bin/python3
from sys import argv

from pysat.formula import CNF
from pysat.solvers import Glucose3, Solver

formula = CNF()
formula.append([-1, 2, 4])
formula.append([1, -2, 5])
formula.append([-1, -2, 6])
formula.append([1, 2, 7])

g = Glucose3(bootstrap_with=formula.clauses)

if g.solve(assumptions=[-4, -5, -6, -7]) == False:
    print("Core: ", g.get_core())
```

Example: naive (deletion) MUS extraction

Input : Set $\mathcal{F}$
Output: Minimal subset $\mathcal{M}$
begin
  $\mathcal{M} \leftarrow \mathcal{F}$
  foreach $c \in \mathcal{M}$ **do**
    if $\neg\mathrm{SAT}(\mathcal{M} \setminus \{c\})$ **then**
      $\mathcal{M} \leftarrow \mathcal{M} \setminus \{c\}$ // If $\neg\mathrm{SAT}(\mathcal{M} \setminus \{c\})$, then $c \notin$ MUS
  return $\mathcal{M}$ // Final $\mathcal{M}$ is MUS
end

- Number of predicate tests: $\mathcal{O}(m)$ [CD91, BDTW93]

Naive MUS extraction I

```
def main():
    cnf = CNF(from_file=argv[1])   # create a CNF object from file
    (rnv, assumps) = add_assumps(cnf)

    oracle = Solver(name='g3', bootstrap_with=cnf.clauses)

    mus = find_mus(assumps, oracle)
    mus = [ref - rnv for ref in mus]
    print("MUS: ", mus)

if __name__== "__main__":
  main()
```

Naive MUS extraction II

```python
def add_assumps(cnf):
    rnv = topv = cnf.nv
    assumps = []                        # list of assumptions to use
    for i in range(len(cnf.clauses)):
        topv += 1
        assumps.append(topv)            # register literal
        cnf.clauses[i].append(-topv)    # extend clause with literal
    cnf.nv = cnf.nv + len(assumps)      # update # of vars
    return rnv, assumps


def main():
    cnf = CNF(from_file=argv[1])    # create a CNF object from file
    (rnv, assumps) = add_assumps(cnf)

    oracle = Solver(name='g3', bootstrap_with=cnf.clauses)

    mus = find_mus(assumps, oracle)
    mus = [ref - rnv for ref in mus]
    print("MUS: ", mus)

if __name__== "__main__":
  main()
```

Naive MUS extraction III

```python
from sys import argv

from pysat.formula import CNF
from pysat.solvers import Solver


def find_mus(assmp, oracle):
    i = 0
    while i < len(assmp):
        ts = assmp[:i] + assmp[(i+1):]
        if not oracle.solve(assumptions=ts):
            assmp = ts
        else:
            i += 1
    return assmp
```

Naive MUS extraction III

```
from sys import argv

from pysat.formula import CNF
from pysat.solvers import Solver


def find_mus(assmp, oracle):
    i = 0
    while i < len(assmp):
        ts = assmp[:i] + assmp[(i+1):]
        if not oracle.solve(assumptions=ts):
            assmp = ts
        else:
            i += 1
    return assmp
```

Demo

A less naive MUS extractor

```
def clset_refine(assmp, oracle):
    assmp = sorted(assmp)
    while True:
        oracle.solve(assumptions=assmp)
        ts = sorted(oracle.get_core())
        if ts == assmp:
            break
        assmp = ts
    return assmp
# ...

def main():
    cnf = CNF(from_file=argv[1])   # create a CNF object from file
    (rnv, assumps) = add_assumps(cnf)

    oracle = Solver(name='g3', bootstrap_with=cnf.clauses)

    assumps = clset_refine(assumps, oracle)
    mus = find_mus(assumps, oracle)
    mus = [ref - rnv for ref in mus]
    print("MUS: ", mus)

if __name__== "__main__":
  main()
```

3 A Glimpse of the Future

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Some final notes

- SAT is a low-level, but very powerful problem solving paradigm
  - PySAT suggests a way to tackle this drawback, but there are others
- There is an ongoing revolution on problem solving with SAT (and SMT) oracles
  - E.g. QBF, model-based diagnosis, explainability, theorem proving, program synthesis, ...
- The use of SAT oracles is impacting problem solving for many different complexity classes
  - With well-known representative problems, e.g. QBF, #SAT, etc.

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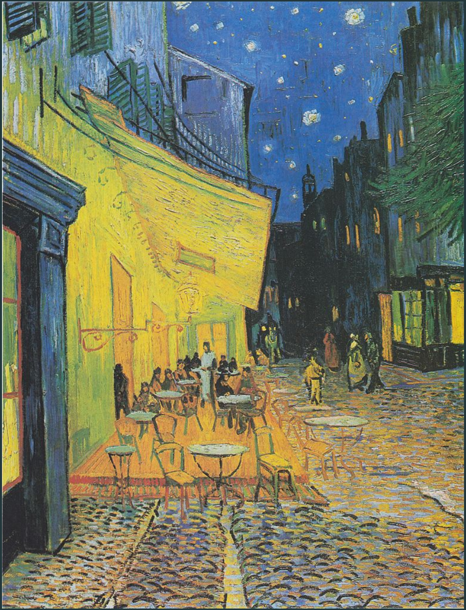
- The use of SAT oracles is impacting problem solving for many different complexity classes
  - With well-known representative problems, e.g. QBF, #SAT, etc.
- Many fascinating research topics out there!
  - Connections with ML seem unavoidable

Sample of tools

- PySAT
- SAT solvers:
  - MiniSat
  - Glucose
- MaxSAT solvers:
  - RC2
  - MSCG
  - OpenWBO
  - MaxHS
- MUS extractors:
  - MUSer
- MCS extractors:
  - mcsXL
  - LBX
  - MCSls
- Many other tools available from the ReasonLab server

Questions?



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