Introduction to SMT

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Overview of the talk

- Motivation
- SMT
- Theories of Interest
- History of SMT
- Eager approach
- Lazy approach
  - Optimizations and DPLL(T)
  - Theory solvers: difference logic and case splitting
  - Combining Theory Solvers
- Limitations and Other Approaches

Introduction

- Originally, automated reasoning $\equiv$ uniform proof-search procedures for FO logic
- Limited success: is FO logic the best compromise between expressivity and efficiency?
- Another trend [Sha02] is to gain efficiency by:
  - addressing only (expressive enough) decidable fragments of a certain logic
  - incorporate domain-specific reasoning, e.g:
    - arithmetic reasoning
    - equality
    - data structures (arrays, lists, stacks, ...)

Introduction (2)

Examples of this alternative trend:

- SAT: use propositional logic as the formalization language
  - \+ high degree of efficiency
  - \- expressive (all NP-complete) but involved encodings
- SMT: propositional logic + domain-specific reasoning
  - \+ improves the expressivity
  - \- certain (but acceptable) loss of efficiency

GOAL OF THIS TALK:
introduce SMT, with its main techniques

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Need and Applications of SMT

- Some problems are more naturally expressed in other logics than propositional logic, e.g:
  - Software verification needs reasoning about equality, arithmetic, data structures, ...
- SMT consists of deciding the satisfiability of a (ground) FO formula with respect to a background theory
- Example (Equality with Uninterpreted Functions – EUF):

$$g(a)=c \quad \wedge \quad (\, f(g(a)) \neq f(c) \vee g(a)=d \,) \quad \wedge \quad c \neq d$$

- Wide range of applications:
  - Predicate abstraction [LNO06]
  - Model checking[AMP06]
  - Scheduling [BNO$^+$08b]
  - Test generation[TdH08]
  - ...

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Theories of Interest - EUF [BD94, NO80, NO07]

- Equality with Uninterpreted Functions, i.e. "$=$" is equality
- If background logic is FO with equality, EUF is empty theory
- Consider formula

  $$a * (f(b) + f(c)) = d \quad \wedge \quad b * (f(a) + f(c)) \neq d \quad \wedge \quad a = b$$

- Formula is UNSAT, but no arithmetic resoning is needed
- If we abstract the formula into

  $$h(a, g(f(b), f(c))) = d \quad \wedge \quad h(b, g(f(a), f(c))) \neq d \quad \wedge \quad a = b$$

  it is still UNSAT
- EUF is used to abstract non-supported constructions, e.g:
  - Non-linear multiplication
  - ALUs in circuits

Theories of Interest - Arithmetic

- Very useful for obvious reasons
- Restricted fragments support more efficient methods:
  - Bounds: $x \bowtie k$ with $\bowtie \in \{<, >, \leq, \geq, =\}$
  - Difference logic: $x - y \bowtie k$, with $\bowtie \in \{<, >, \leq, \geq, =\}$ [NO05, WIGG05, SM06]
  - UTVPI: $\pm x \pm y \bowtie k$, with $\bowtie \in \{<, >, \leq, \geq, =\}$ [LM05]
  - Linear arithmetic, e.g: $2x - 3y + 4z \leq 5$ [DdM06]
  - Non-linear arithmetic, e.g: $2xy + 4xz^2 - 5y \leq 10$ [BLNM$^+$09, ZM10]
  - Variables are either reals or integers
- Machine-inspired arithmetic: floating-point arithmetic

Th. of Int.- Arrays[SBDL01, BNO$^+$08a, dMB09]

- Two interpreted function symbols *read* and *write*
- Theory is axiomatized by:
  - $\forall a \forall i \forall v\ (read(write(a,i,v),i) = v)$
  - $\forall a \forall i \forall j \forall v\ (i \neq j \rightarrow read(write(a,i,v),j) = read(a,j))$
- Sometimes extensionality is added:
  - $\forall a \forall b\ (\ (\forall i (read(a,i) = read(b,i))\ ) \rightarrow a = b$

- Is the following set of literals satisfiable?

$$\begin{array}{ccc} write(a,i,x) \neq b & read(b,i) = y & read(write(b,i,x),j) = y \\ a = b & & i = j \end{array}$$

- Used for:
  - Software verification
  - Hardware verification (memories)

Th. of Interest - Bit vectors [BCF$^+$07, BB09]

- Constants represent vectors of bits
- Useful both for hardware and software verification
- Different type of operations:
  - String-like operations: concat, extract, ...
  - Logical operations: bit-wise not, or, and, ...
  - Arithmetic operations: add, substract, multiply, ...
- Assume bit-vectors have size 3. Is the formula SAT?

$$a[0:1] \neq b[0:1] \ \wedge \ (a|b) = c \ \wedge \ c[0] = 0 \ \wedge \ a[1] + b[1] = 0$$

Combina. of theories [NO79, Sho84, BBC$^+$05]

- In practice, theories are not isolated
- Software verifications needs arithmetic, arrays, bitvectors, ...
- Formulas of the following form usually arise:

$$a = b + 2 \land A = write(B, a + 1, 4) \land (read(A, b + 3) = 2 \lor f(a - 1) \neq f(b + 1))$$

- The goal is to combine decision procedures for each theory

SMT in Practice

GOOD NEWS: efficient decision procedures for sets of ground literals exist for various theories of interest

PROBLEM: in practice, we need to deal with:

(1) arbitrary Boolean combinations of literals ($\wedge, \vee, \neg$)
(DNF conversion is not a solution in practice)

(2) multiple theories

(3) quantifiers

We will only focus on (1) and (2), but techniques for (3) exist.

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SMT Prehistory - Late 70's and 80's

- Pioneers:
  - R. Boyer, J. Moore, G. Nelson, D. Open, R. Shostak
- Influential results:
  - Nelson-Oppen congruence closure procedure [NO80]
  - Nelson-Oppen combination method [NO79]
  - Shostak combination method [Sho84]
- Influential systems:
  - Nqthm prover [BM90] [Boyer, Moore]
  - Simplify [DNS05] [Detlefs, Nelson, Saxe]

Beginnings of SMT - Early 2000s

KEY FACT: SAT solvers improved performance

Two ways of exploiting this fact:

- Eager approach: encode SMT into SAT

  [Bryant, Lahiri, Pnueli, Seshia, Strichman, Velev, ...]

  [PRSS99, SSB02, SLB03, BGV01, BV02]

- Lazy approach: plug SAT solver with a decision procedure

  [Armando, Barrett, Castellini, Cimatti, Dill, Giunchiglia, deMoura, Ruess, Sebastiani, Stump,...]

  [ACG00, dMR02, BDS02a, ABC$^+$02]

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Eager approach

- Methodology: translate problem into equisatisfiable propositional formula and use off-the-shelf SAT solver
- Why "eager"?
  Search uses all theory information from the beginning
- Characteristics:
  - \+ Can use best available SAT solver
  - \- Sophisticated encodings are needed for each theory

Eager approach – Example

Let us consider an EUF formula:

- First step: remove function/predicate symbols. Assume we have terms $f(a)$, $f(b)$ and $f(c)$.
  - Ackermann reduction:
    - Replace them by fresh constants A, B and C
    - Add clauses:
      $$\begin{array}{lcl} a=b & \rightarrow & A=B \\ a=c & \rightarrow & A=C \\ b=c & \rightarrow & B=C \end{array}$$
  - Bryant reduction:
    - Replace $f(a)$ by A
    - Replace $f(b)$ by $ite(b=a, A, B)$
    - Replace $f(c)$ by $ite(c=a, A, ite(c=b, B, C))$

Now, atoms are equalities between **constants**

Eager approach – Example (2)

- Second step: encode formula into propositional logic
  - Small-domain encoding:
    - If there are n different constants, there is a model with size at most n
    - $\log n$ bits to encode the value of each constant
    - $a = b$ translated using the bits for a and b
  - Per-constraint encoding:
    - Each atom $a = b$ is replaced by var $P_{a,b}$
    - Transitivity constraints are added (e.g. $P_{a,b} \wedge P_{b,c} \rightarrow P_{a,c}$)

This is a **very rough** overview of an encoding from EUF to SAT.

See [PRSS99, SSB02, SLB03, BGV01, BV02] for details.

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Lazy approach

Methodology:

Example: consider EUF and the CNF

$$\underbrace{g(a)=c}_{1} \quad \wedge \quad (\underbrace{f(g(a))\neq f(c)}_{\overline{2}} \vee \underbrace{g(a)=d}_{3}) \quad \wedge \quad \underbrace{c\neq d}_{\overline{4}}$$

- SAT solver returns model $[1, \overline{2}, \overline{4}]$

Lazy approach

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- Send $\{1, \overline{2} \vee 3, \overline{4}, \overline{1} \vee 2 \vee 4\}$ to SAT solver
- SAT solver returns model $[1, 2, 3, \overline{4}]$
- Theory solver says T-inconsistent
- SAT solver detects $\{1, \overline{2} \vee 3, \overline{4}, \overline{1} \vee 2 \vee 4, \overline{1} \vee \overline{2} \vee \overline{3} \vee 4\}$
  UNSATISFIABLE

Lazy approach (2)

- Why “lazy”?

  Theory information used lazily when checking T-consistency of propositional models

- Characteristics:

  + Modular and flexible
  - Theory information does not guide the search

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Lazy approach - Optimizations

Several optimizations for enhancing efficiency:

- Check T-consistency only of full propositional models

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- Check T-consistency of partial assignment while being built

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- Given a T-inconsistent assignment M, add $\neg M$ as a clause

Lazy approach - Optimizations

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- Given a T-inconsistent assignment M, identify a T-inconsistent subset $M_0 \subseteq M$ and add $\neg M_0$ as a clause

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- Upon a T-inconsistency, add clause and restart

Lazy approach - Optimizations

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- Given a T-inconsistent assignment M, identify a T-inconsistent subset $M_0 \subseteq M$ and add $\neg M_0$ as a clause

- ~~Upon a T-inconsistency, add clause and restart~~
- Upon a T-inconsistency, bactrack to some point where the assignment was still T-consistent

Lazy approach - T-propagation

- As pointed out the lazy approach has one drawback:
  - Theory information does not guide the search (too lazy)
- How can we improve that? For example:

  Assume that $a < b$, $b < c$ are in our partial assignment M.

  If the formula contains $a < c$ we would like to add it to M
- Search guided by T-Solver by finding T-consequences, instead of only validating it as in basic lazy approach.
- Naive implementation::
  Add $\neg l$. If T-inconsistent then infer l [ACG00]
  But for efficient Theory Propagation we need:

  - T-Solvers specialized and fast in it.
  - fully exploited in conflict analysis
- This approach has been named DPLL(T) [NOT06]

Lazy approach - Important points

Important and benefitial aspects of the lazy approach:
(even with the optimizations)

- Everyone does what he/she is good at:
  - SAT solver takes care of Boolean information
  - Theory solver takes care of theory information
- Theory solver only receives conjunctions of literals
- Modular approach:
  - SAT solver and T-solver communicate via a simple API
  - SMT for a new theory only requires new T-solver
  - SAT solver can be embedded in a lazy SMT system with relatively litte effort

DPLL(T)

In a nutshell:

$$\text{DPLL}(T) = \text{DPLL}(X) + T\text{-Solver}$$

- DPLL(X):
  - Very similar to a SAT solver, enumerates Boolean models
  - Not allowed: pure literal, blocked literal detection, ...
  - Desirable: partial model detection
- T-Solver:
  - Checks consistency of conjunctions of literals
  - Computes theory propagations
  - Produces explanations of inconsistency/T-propagation
  - Should be incremental and backtrackable

DPLL(T) - Example

Consider again EUF and the formula:

$$\underbrace{g(a)=c}_{1} \;\wedge\; (\underbrace{f(g(a))\neq f(c)}_{\overline{2}} \vee \underbrace{g(a)=d}_{3}) \;\wedge\; \underbrace{c\neq d}_{\overline{4}}$$

$$\emptyset \;\|\; 1,\; \overline{2}\vee 3,\; \overline{4} \quad\Rightarrow\quad (\text{UnitPropagate})$$

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$$\begin{array}{rcll} \emptyset \;\|\; & 1,\; \overline{2}\vee 3,\; \overline{4} & \Rightarrow & (\text{UnitPropagate}) \\ 1 \;\|\; & 1,\; \overline{2}\vee 3,\; \overline{4} & \Rightarrow & (\text{UnitPropagate}) \\ 1\,\overline{4} \;\|\; & 1,\; \overline{2}\vee 3,\; \overline{4} & \Rightarrow & (\text{T-Propagate}) \end{array}$$

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1\,\overline{4} & \| & 1,\ \overline{2}\vee 3,\ \overline{4} & \Rightarrow & (\text{T-Propagate}) \\
1\,\overline{4}\,2 & \| & 1,\ \overline{2}\vee 3,\ \overline{4} & \Rightarrow & (\text{T-Propagate})
\end{array}$$

DPLL(T) - Example

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1\ \overline{4}\ 2\ \overline{3} & \parallel & 1,\ \overline{2}\vee 3,\ \overline{4} & \Rightarrow & (\text{Fail})
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DPLL(T) - Example

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\end{array}$$

UNSAT

DPLL(T) - Overall algorithm

High-levew view gives the same algorithm as a CDCL SAT solver:

```
while(true){

    while (propagate_gives_conflict()){
        if (decision_level==0) return UNSAT;
        else analyze_conflict();
    }

    restart_if_applicable();
    remove_lemmas_if_applicable();

    if (!decide()) returns SAT; // All vars assigned
}
```

Differences are in:

- `propagate_gives_conflict`
- `analyze_conflict`

DPLL(T) - Propagation

```
propagate_gives_conflict( ) returns Bool

    do {

      // unit propagate
      if ( unit_prop_gives_conflict() ) then return true

      // check T-consistency of the model
      if ( solver.is_model_inconsistent() ) then return true

      // theory propagate
      solver.theory_propagate()

    } while (someTheoryPropagation)

    return false
```

DPLL(T) - Propagation (2)

- Three operations:
  - Unit propagation (SAT solver)
  - Consistency checks (T-solver)
  - Theory propagation (T-solver)
- Cheap operations are computed first
- If theory is expensive, calls to T-solver are sometimes skipped
- For completeness, only necessary to call T-solver at the leaves (i.e. when we have a full propositional model)
- Theory propagation is not necessary for completeness

DPLL(T) - Conflict Analysis

Remember conflict analysis in SAT solvers:

```
$C$:= conflicting clause
while $C$ contains more than one lit of last DL
    $l$:=last literal assigned in $C$
    $C$:=Resolution($C$,reason($l$))
end while

// let C = C′ v l where l is UIP
backjump(maxDL(C′))
add l to the model with reason $C$
learn(C)
```

DPLL(T) - Conflict Analysis (2)

Conflict analysis in DPLL(T):

```
if boolean conflict then $C$:= conflicting clause
else $C$:=$\neg$( solver.explain_inconsistency() )
while $C$ contains more than one lit of last DL
    $l$:=last literal assigned in $C$
    $C$:=Resolution($C$,reason($l$))
end while

// let C = C′ v l where l is UIP
backjump(maxDL(C′))
add l to the model with reason $C$
learn(C)
```

DPLL(T) - Conflict Analysis (3)

What does `explain_inconsistency` return?

- A (small) conjuntion of literals $l_1 \wedge \ldots \wedge l_n$ such that:
  - They were in the model when T-inconsistency was found
  - It is T-inconsistent

What is now $reason(l)$?

- If l was unit propagated, reason is the clause that propagated it
- If l was T-propagated?
  - T-solver has to provide an explanation for l, i.e. a (small) set of literals $l_1, \ldots, l_n$ such that:
    - They were in the model when l was T-propagated
    - $l_1 \wedge \ldots \wedge l_n \models_T l$
  - Then $reason(l)$ is $\neg l_1 \vee \ldots \vee \neg l_n \vee l$

DPLL(T) - Conflict Analysis (4)

Let M be of the form $\ldots, c = b, \ldots$ and let F contain

$$h(a) = h(c) \;\vee\; p \qquad a = b \;\vee\; \neg p \;\vee\; a = d \qquad a \neq d \;\vee\; a = b$$

Take the following sequence:

1. Decide $h(a) \neq h(c)$
2. UnitPropagate p (due to clause $h(a) = h(c) \;\vee\; p$)
3. T-Propagate $a \neq b$ (since $h(a) \neq h(c)$ and $c = b$)
4. UnitPropagate $a = d$ (due to clause $a = b \;\vee\; \neg p \;\vee\; a = d$)
5. Conflicting clause $a \neq d \;\vee\; a = b$

Explain($a \neq b$) is $\{h(a) \neq h(c), c = b\}$

$$\dfrac{h(a) = h(c) \vee \mathbf{p} \qquad \dfrac{\overset{\big\downarrow}{h(a) = h(c) \vee c \neq b \vee \mathbf{a \neq b}} \qquad \dfrac{a = b \vee \neg p \vee \mathbf{a = d} \quad \mathbf{a \neq d} \vee a = b}{\mathbf{a = b} \vee \neg p}}{h(a) = h(c) \vee c \neq b \vee \neg \mathbf{p}}}{h(a) = h(c) \;\vee\; c \neq b}$$

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Difference logic

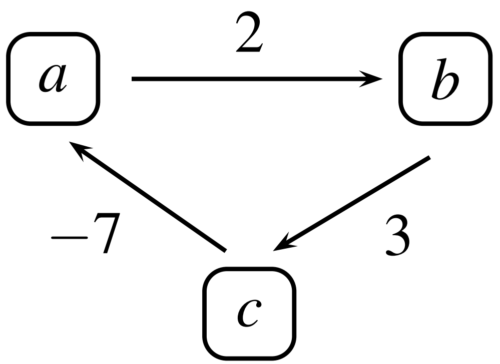
- Literals in Difference Logic are of the form $a - b \bowtie k$, where
  - $\bowtie \in \{\leq,\ \geq,\ <,\ >,\ =,\ \neq\}$
  - a and b are integer/real variables
  - k is an integer/real
- At the formula level,
  $a = b$ is replaced by p and $p \leftrightarrow a \leq b \wedge b \leq a$ is added
- If domain is $\mathbb{Z}$ then $a - b < k$ is replaced by $a - b \leq k - 1$
- If domain is $\mathbb{R}$ then $a - b < k$ is replaced by $a - b \leq k - \delta$
  - δ is a sufficiently small real
  - δ is not computed but used symbolically
    (i.e. numbers are pairs (k, δ))
- Hence we can assume all literals are $a - b \leq k$

Difference Logic - Remarks

- Note that any solution to a set of DL literals can be shifted
  (i.e. if σ is a solution then $\sigma'(x) = \sigma(x) + k$ also is a solution)
- This allows one to process bounds $x \leq k$
  - Introduce fresh variable $zero$
  - Convert all bounds $x \leq k$ into $x - zero \leq k$
  - Given a solution σ, shift it so that $\sigma(zero) = 0$
- If we allow (dis)equalities as literals, then:
  - If domain is $\mathbb{R}$ consistency check is polynomial
  - If domain is $\mathbb{Z}$ consistency check is NP-hard
    (k-colorability)
    - $1 \leq c_i \leq k$ with $i = 1 \ldots \#verts$ encodes k colors available
    - $c_i \neq c_j$ if i and j adjacents encode proper assignment

Difference Logic as a Graph Problem

- Given $M = \{a - b \leq 2,\ b - c \leq 3,\ c - a \leq -7\}$, construct weighted graph $\mathcal{G}(M)$



- **Theorem:**

  M is T-inconsistent iff $\mathcal{G}(M)$ has a negative cycle

Difference Logic as a Graph Problem (2)

Theorem:

M is T-inconsistent iff $\mathcal{G}(M)$ has a negative cycle

$\Leftarrow)$

Any negative cycle $a_1 \xrightarrow{k_1} a_2 \xrightarrow{k_2} a_3 \longrightarrow \ldots \longrightarrow a_n \xrightarrow{k_n} a_1$ corresponds to a set of literals:

$$\begin{array}{l} a_1 - a_2 \leq k_1 \\ a_2 - a_3 \leq k_2 \\ \ldots \\ a_n - a_1 \leq k_n \end{array}$$

If we add them all, we get $0 \leq k_1 + k_2 + \ldots + k_n$, which is inconsistent since neg. cycle implies $k_1 + k_2 + \ldots + k_n < 0$

Difference Logic as a Graph Problem (3)

Theorem:

M is T-inconsistent iff $\mathcal{G}(M)$ has a negative cycle

$\Rightarrow)$
Let us assume that there is no negative cycle.

1. Consider additional vertex o with edges $o \xrightarrow{0} v$ to all verts. v
2. For each variable x, let $\sigma(x) = -dist(o,x)$
   [exists because there is no negative cycle]
3. σ is a model of M
   - If $\sigma \not\models x - y \leq k$ then $-dist(o,x) + dist(o,y) > k$
   - Hence, $dist(o,y) > dist(o,x) + k$
   - But $k = weight(x \longrightarrow y)$!!!

Bellman-Ford: negative cycle detection

```
forall $v \in V$ do $d[v] := \infty$ endfor
forall $i = 1$ to $|V| - 1$ do
    forall $(u, v) \in E$ do
        if d[v] >  d[u] + weight(u,v) then
            d[v]:= d[u] + weight(u,v)
            p[v]:= u
      endif
    endfor
endfor

forall $(u, v) \in E$ do
    if $d[v] > d[u] + weight(u, v)$ then
        Negative cycle detected
        Cycle reconstructed following $p$
    endif
endfor
```

Consistency checks

- Consistency checks can be performed using Bellman-Ford in time ($O(|V| \cdot |E|)$)
- Other more efficient variants exists
- Incrementality easy:
  - Upon arrival of new literal $a \xrightarrow{k} b$ process graph from u
- Solutions can be kept after backtracking
- Inconsistency explanations are negative cycles
  (irredundant but not minimal explanations)

Theory propagation

- Addition of $a \xrightarrow{k} b$ entails $c - d \leq k'$ only if

$$\underbrace{c \longrightarrow\ast\ \mathrlap{\overbrace{\phantom{a \xrightarrow{k} b \longrightarrow\ast\ d}}^{shortest}} a \xrightarrow{k} b}_{shortest} \longrightarrow\ast\ d$$

- Given a solution σ, each edge $a \xrightarrow{k} b$ (i.e. $a - b \leq k$) has its reduced cost $k - \sigma(a) + \sigma(b) \geq 0$
- Shortest path computation more efficient using reduced costs, since they are non-negative [Dijkstra's algorithm]
- Theory propagation ≈ shortest-path computations
- Explanations are the shortest paths

Overview of the talk

- Motivation
- SMT
- Theories of Interest
- Eager approach
- Lazy approach
  - Optimizations and DPLL(T)
  - T-solvers: case splitting
  - Combining Theory Solvers
- Limitations and Other Approaches

Case Reasoning in Theory Solvers

- For certain theories, consistency checking requires case reasoning.
- Example: consider the theory of arrays and the set of literals

  $$read(write(A,i,x),j) \neq x \qquad read(write(A,i,x),j) \neq read(A,j)$$

  Two cases:

  - $i = j$. LHS rewrites into $x \neq x$!!!
  - $i \neq j$. RHS rewrites into $read(A,j) \neq read(A,j)$!!!

CONCLUSION: T-inconsistent

Case Reasoning in Theory Solvers (2)

- A complete T-solver reasons by cases via internal case splitting and backtracking mechanisms.
- An alternative is to lift case splitting and backtracking from the T-solver to the SAT engine.
- Basic idea: encode case splits as sets of clauses and send them as needed to the SAT engine for it to split on them.
- Possible benefits:
  - All case-splitting is coordinated by the SAT engine
  - Only have to implement case-splitting infrastructure in one place
  - Can learn a wider class of lemmas (more details later)

Case Reasoning in Theory Solvers (3)

- Basic idea: encode case splits as a set of clauses and send them as needed to the SAT engine
- Example:
  - Assume model contains literal $s = \underbrace{read(write(A,i,t),j)}_{s'}$
  - DPLL(X) asks: "is it T-satisfiable"?
  - T-solver says: "I do not know **yet**, but it will be helpful that you consider these theory lemmas:"

$$s = s' \land i = j \longrightarrow s = t$$

$$s = s' \land i \neq j \longrightarrow s = read(A,j)$$

- We need certain completeness conditions (e.g. once all lits from a certain subset $\mathcal{L}$ has been decided, the T-solver should YES/NO)

Overview of the talk

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  - Theory solvers: difference logic and case splitting
  - Combining Theory Solvers
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Need for combination

- In software verification, formulas like the following one arise:

$$a = b + 2 \wedge A = write(B, a+1, 4) \wedge (read(A, b+3) = 2 \vee f(a-1) \neq f(b+1))$$

- Here reasoning is needed over
  - The theory of linear arithmetic ($\mathbb{T}_{LA}$)
  - The theory of arrays ($\mathbb{T}_{A}$)
  - The theory of uninterpreted functions ($\mathbb{T}_{EUF}$)

- Remember that T-solvers only deal with conjunctions of lits.

- Given T-solvers for the three individual theories, can we combine them to obtain one for ($\mathbb{T}_{LA} \cup \mathbb{T}_{A} \cup \mathbb{T}_{EUF}$)?

- Under certain conditions the **Nelson-Oppen** combination method gives a positive answer

Motivating example - Convex case

Consider the following set of literals:

$$\begin{array}{rcl} f(f(x)-f(y)) & = & a \\ f(0) & = & a+2 \\ x & = & y \end{array}$$

There are two theories involved: $\mathbb{T}_{LA(\mathbb{R})}$ and $\mathbb{T}_{EUF}$

FIRST STEP: purify each literal so that it belongs to a single theory

$$f(f(x)-f(y)) = a \implies \begin{array}{rcl} f(e_1) & = & a \\ e_1 & = & f(x)-f(y) \end{array} \implies \begin{array}{rcl} f(e_1) & = & a \\ e_1 & = & e_2-e_3 \\ e_2 & = & f(x) \\ e_3 & = & f(y) \end{array}$$

Motivating example - Convex case

Consider the following set of literals:

$$\begin{array}{rcl} f(f(x)-f(y)) & = & a \\ f(0) & = & a+2 \\ x & = & y \end{array}$$

There are two theories involved: $\mathbb{T}_{LA(\mathbb{R})}$ and $\mathbb{T}_{EUF}$

FIRST STEP: purify each literal so that it belongs to a single theory

$$\begin{array}{rcl} f(0) & = & a+2 \end{array} \Longrightarrow \begin{array}{rcl} f(e_4) & = & a+2 \\ e_4 & = & 0 \\ & & \end{array} \Longrightarrow \begin{array}{rcl} f(e_4) & = & e_5 \\ e_4 & = & 0 \\ e_5 & = & a+2 \end{array}$$

Motivating example - Convex case (2)

SECOND STEP: check satisfiability and exchange entailed equalities

$$\begin{array}{lcl@{\qquad\qquad}lcr}
& EUF & & & Arithmetic & \\
f(e_1) & = & a & e_2 - e_3 & = & e_1 \\
f(x) & = & e_2 & e_4 & = & 0 \\
f(y) & = & e_3 & e_5 & = & a+2 \\
f(e_4) & = & e_5 & & & \\
x & = & y & & &
\end{array}$$

The two solvers only share constants: $e_1, e_2, e_3, e_4, e_5, a$

To merge the two models into a single one, the solvers have to agree on equalities between shared constants (interface equalities)

This can be done by exchanging entailed interface equalities

Motivating example - Convex case (2)

SECOND STEP: check satisfiability and exchange entailed equalities

EUF	$Arithmetic$
$f(e_1) = a$	$e_2 - e_3 = e_1$
$f(x) = e_2$	$e_4 = 0$
$f(y) = e_3$	$e_5 = a + 2$
$f(e_4) = e_5$	$e_2 = e_3$
$x = y$	

The two solvers only share constants: $e_1, e_2, e_3, e_4, e_5, a$

- EUF-Solver says SAT
- Ari-Solver says SAT
- $EUF \models e_2 = e_3$

Motivating example - Convex case (2)

SECOND STEP: check satisfiability and exchange entailed equalities

EUF			$Arithmetic$		
$f(e_1)$	$=$	a	$e_2 - e_3$	$=$	e_1
$f(x)$	$=$	e_2	e_4	$=$	0
$f(y)$	$=$	e_3	e_5	$=$	$a+2$
$f(e_4)$	$=$	e_5	e_2	$=$	e_3
x	$=$	y			
e_1	$=$	e_4			

The two solvers only share constants: $e_1, e_2, e_3, e_4, e_5, a$

- EUF-Solver says SAT
- Ari-Solver says SAT
- $Ari \models e_1 = e_4$

Motivating example - Convex case (2)

SECOND STEP: check satisfiability and exchange entailed equalities

EUF			$Arithmetic$		
$f(e_1)$	$=$	a	$e_2 - e_3$	$=$	e_1
$f(x)$	$=$	e_2	e_4	$=$	0
$f(y)$	$=$	e_3	e_5	$=$	$a+2$
$f(e_4)$	$=$	e_5	e_2	$=$	e_3
x	$=$	y	a	$=$	e_5
e_1	$=$	e_4			

The two solvers only share constants: $e_1, e_2, e_3, e_4, e_5, a$

- EUF-Solver says SAT
- Ari-Solver says SAT
- $EUF \models a = e_5$

Motivating example - Convex case (2)

SECOND STEP: check satisfiability and exchange entailed equalities

$$
\begin{array}{lcr@{\qquad\qquad}lcr}
\multicolumn{3}{c}{EUF} & \multicolumn{3}{c}{Arithmetic} \\
f(e_1) & = & a & e_2 - e_3 & = & e_1 \\
f(x) & = & e_2 & e_4 & = & 0 \\
f(y) & = & e_3 & e_5 & = & a+2 \\
f(e_4) & = & e_5 & e_2 & = & e_3 \\
x & = & y & a & = & e_5 \\
e_1 & = & e_4 & & &
\end{array}
$$

The two solvers only share constants: $e_1, e_2, e_3, e_4, e_5, a$

- EUF-Solver says SAT
- *Ari*-Solver says **UNSAT**
- Hence the original set of lits was **UNSAT**

Nelson-Oppen – The convex case

- A theory T is stably-infinite iff every T-satisfiable quantifier-free formula has an infinite model
- A theory T is convex iff
  $S \models_T a_1 = b_1 \vee \ldots \vee a_n = b_n \implies S \models a_i = b_i$ for some i

Deterministic Nelson-Oppen: [NO79, TH96, MZ02]

- Given two signature-disjoint, stably-infinite and convex theories T_1 and T_2
- Given a set of literals S over the signature of $T_1 \cup T_2$
- The $(T_1 \cup T_2)$-satisfiability of S can be checked with the following algorithm:

Nelson-Oppen – The convex case (2)

Deterministic Nelson-Oppen

1. Purify S and split it into $S_1 \cup S_2$.
   Let $\mathcal{E}$ the set of interface equalities between S_1 and S_2
2. If S_1 is T_1-unsatisfiable then **UNSAT**
3. If S_2 is T_2-unsatisfiable then **UNSAT**
4. If $S_1 \models_{T_1} x = y$ with $x = y \in \mathcal{E} \setminus S_2$ **then**
   $S_2 := S_2 \cup \{x = y\}$ and **goto** 3
5. If $S_2 \models_{T_2} x = y$ with $x = y \in \mathcal{E} \setminus S_1$ **then**
   $S_1 := S_1 \cup \{x = y\}$ and **goto** 2
6. Report **SAT**

Motivating example – Non-convex case

Consider the following **UNSATISFIABLE** set of literals:

$$\begin{array}{rcl} 1 \leq & x & \leq 2 \\ f(1) & = & a \\ f(x) & = & b \\ a & = & b+2 \\ f(2) & = & f(1)+3 \end{array}$$

There are two theories involved: $\mathbb{T}_{LA(\mathbb{Z})}$ and $\mathbb{T}_{EUF}$

FIRST STEP: purify each literal so that it belongs to a single theory

$$f(1) = a \implies \begin{array}{rcl} f(e_1) & = & a \\ e_1 & = & 1 \end{array}$$

Motivating example – Non-convex case

Consider the following **UNSATISFIABLE** set of literals:

$$\begin{array}{rcl} 1 \leq & x & \leq 2 \\ f(1) & = & a \\ f(x) & = & b \\ a & = & b+2 \\ f(2) & = & f(1)+3 \end{array}$$

There are two theories involved: $\mathbb{T}_{LA(\mathbb{Z})}$ and $\mathbb{T}_{EUF}$

FIRST STEP: purify each literal so that it belongs to a single theory

$$f(2) = f(1)+3 \quad \Longrightarrow \quad \begin{array}{rcl} e_2 & = & 2 \\ f(e_2) & = & e_3 \\ f(e_1) & = & e_4 \\ e_3 & = & e_4+3 \end{array}$$

Motivating example – Non-convex case(2)

SECOND STEP: check satisfiability and exchange entailed equalities

$Arithmetic$	EUF
$1 \leq x$	$f(e_1) = a$
$x \leq 2$	$f(x) = b$
$e_1 = 1$	$f(e_2) = e_3$
$a = b+2$	$f(e_1) = e_4$
$e_2 = 2$	
$e_3 = e_4 + 3$	
$a = e_4$	

The two solvers only share constants: $x, e_1, a, b, e_2, e_3, e_4$

- Ari-Solver says SAT
- EUF-Solver says SAT
- $EUF \models a = e_4$

Motivating example – Non-convex case(2)

SECOND STEP: check satisfiability and exchange entailed equalities

$$\begin{array}{rcl} & Arithmetic & \\ 1 & \leq & x \\ x & \leq & 2 \\ e_1 & = & 1 \\ a & = & b+2 \\ e_2 & = & 2 \\ e_3 & = & e_4+3 \\ a & = & e_4 \end{array} \qquad \begin{array}{rcl} & EUF & \\ f(e_1) & = & a \\ f(x) & = & b \\ f(e_2) & = & e_3 \\ f(e_1) & = & e_4 \end{array}$$

The two solvers only share constants: $x, e_1, a, b, e_2, e_3, e_4$

- Ari-Solver says SAT
- EUF-Solver says SAT
- No theory entails any other interface equality, but...

Motivating example – Non-convex case(2)

SECOND STEP: check satisfiability and exchange entailed equalities

$$\begin{array}{rcl@{\qquad\qquad}rcl}
\multicolumn{3}{c}{Arithmetic} & \multicolumn{3}{c}{EUF} \\
1 & \leq & x & f(e_1) & = & a \\
x & \leq & 2 & f(x) & = & b \\
e_1 & = & 1 & f(e_2) & = & e_3 \\
a & = & b+2 & f(e_1) & = & e_4 \\
e_2 & = & 2 & & & \\
e_3 & = & e_4+3 & & & \\
a & = & e_4 & & &
\end{array}$$

The two solvers only share constants: $x, e_1, a, b, e_2, e_3, e_4$

- Ari-Solver says SAT
- EUF-Solver says SAT
- $Ari \models_T x = e_1 \vee x = e_2$. Let's consider both cases.

Motivating example – Non-convex case(2)

SECOND STEP: check satisfiability and exchange entailed equalities

$Arithmetic$	EUF
$1 \leq x$	$f(e_1) = a$
$x \leq 2$	$f(x) = b$
$e_1 = 1$	$f(e_2) = e_3$
$a = b+2$	$f(e_1) = e_4$
$e_2 = 2$	$x = e_1$
$e_3 = e_4+3$	
$a = e_4$	
$x = e_1$	

- Ari-Solver says SAT
- EUF-Solver says SAT
- $EUF \models_T a = b$, that when sent to Ari makes it **UNSAT**

Motivating example – Non-convex case(2)

SECOND STEP: check satisfiability and exchange entailed equalities

$$
\begin{array}{ccc}
 & \mathit{Arithmetic} & \\
1 & \le & x \\
x & \le & 2 \\
e_1 & = & 1 \\
a & = & b+2 \\
e_2 & = & 2 \\
e_3 & = & e_4+3 \\
a & = & e_4
\end{array}
\qquad
\begin{array}{ccc}
 & \mathit{EUF} & \\
f(e_1) & = & a \\
f(x) & = & b \\
f(e_2) & = & e_3 \\
f(e_1) & = & e_4
\end{array}
$$

Let’s try now with $x = e_2$

Motivating example – Non-convex case(2)

SECOND STEP: check satisfiability and exchange entailed equalities

$$\begin{array}{rcl@{\qquad\qquad}rcl} \multicolumn{3}{c}{Arithmetic} & \multicolumn{3}{c}{EUF} \\ 1 & \leq & x & f(e_1) & = & a \\ x & \leq & 2 & f(x) & = & b \\ e_1 & = & 1 & f(e_2) & = & e_3 \\ a & = & b+2 & f(e_1) & = & e_4 \\ e_2 & = & 2 & x & = & e_2 \\ e_3 & = & e_4+3 & & & \\ a & = & e_4 & & & \\ x & = & e_2 & & & \end{array}$$

- *Ari*-Solver says SAT
- EUF-Solver says SAT
- $EUF \models_T b = e_3$, that when sent to *Ari* makes it **UNSAT**

Motivating example – Non-convex case(2)

SECOND STEP: check satisfiability and exchange entailed equalities

$$\begin{array}{rcr}
 & Arithmetic & \\
1 & \leq & x \\
x & \leq & 2 \\
e_1 & = & 1 \\
a & = & b+2 \\
e_2 & = & 2 \\
e_3 & = & e_4+3 \\
a & = & e_4 \\
x & = & e_2
\end{array}
\qquad
\begin{array}{lcr}
 & EUF & \\
f(e_1) & = & a \\
f(x) & = & b \\
f(e_2) & = & e_3 \\
f(e_1) & = & e_4 \\
x & = & e_2
\end{array}$$

Since both $x = e_1$ and $x = e_2$ are **UNSAT**, the set of literals is **UNSAT**

Nelson-Oppen - The non-convex case

- In the previous example Deterministic NO does not work
- This was because $T_{LA(\mathbb{Z})}$ is not convex:

  $S_{LA(\mathbb{Z})} \models_{T_{LA(\mathbb{Z})}} x = e_1 \vee x = e_2$, but

  $S_{LA(\mathbb{Z})} \not\models_{T_{LA(\mathbb{Z})}} x = e_1$ and

  $S_{LA(\mathbb{Z})} \not\models_{T_{LA(\mathbb{Z})}} x = e_2$
- However, there is a version of NO for non-convex theories
- Given a set constants $\mathcal{C}$, an arrangement $\mathcal{A}$ over $\mathcal{C}$ is:
  - A set of equalities and disequalites between constants in $\mathcal{C}$
  - For each $x, y \in \mathcal{C}$ either $x = y \in \mathcal{A}$ or $x \neq y \in \mathcal{A}$

Nelson-Oppen – The non-convex case (2)

Non-deterministic Nelson-Oppen: [NO79, TH96, MZ02]

- Given two signature-disjoint, stably-infinite theories T_1 and T_2
- Given a set of literals S over the signature of $T_1 \cup T_2$
- The $(T_1 \cup T_2)$-satisfiability of S can be checked via:

1. Purify S and split it into $S_1 \cup S_2$
   Let C be the set of shared constants
2. **For every** arrangement $\mathcal{A}$ over C **do**
   If $(S_1 \cup \mathcal{A})$ is T_1-satisfiable and $(S_2 \cup \mathcal{A})$ is T_2-satisfiable
   report **SAT**
3. Report **UNSAT**

Overview of the talk

- Motivation
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- Eager approach
- Lazy approach
  - Optimizations and DPLL(T)
  - Theory solvers: difference logic and case splitting
  - Combining Theory Solvers
- Limitations and Other Approaches

Eager vs Lazy Approach

REMEMBER....

Important and benefitial aspects of the lazy approach:
(even with the optimizations)

- Everyone does what he/she is good at:
  - SAT solver takes care of Boolean information
  - Theory solver takes care of theory information
- Theory solver only receives conjunctions of literals
- Modular approach:
  - SAT solver and T-solver communicate via a simple API
  - SMT for a new theory only requires new T-solver
  - SAT solver can be embedded in a lazy SMT system with very few new lines of code

Eager vs Lazy Approach (2)

- The Lazy Approach idea (*SAT Solver + Theory Reasoner*) can be applied to other extensions of SAT:
  - Cardinality constraints (e.g. $x_1 + x_2 + \ldots + x_7 \leq 4$)
  - Pseudo-Boolean constraints (e.g. $7x_1 + 4x_2 + 3x_3 + 5x_4 \leq 10$)
  - ...
- Also sophisticated encodings exist for these constraints (Eager Approach)
- Lazy approach seems to dominate, but can we claim that it is always the best option?

Eager vs Lazy Approach (3)

Consider the problem with no SAT clauses and two constraints:

$$\begin{array}{l} x_1 + \ldots + x_n \leq n/2 \\ x_1 + \ldots + x_n > n/2 \end{array}$$

Let us see how a (very) Lazy Approach would behave:

- Problem is obviously unsatisfiable
- Inconsistency explanations are of the form:

Eager vs Lazy Approach (3)

Consider the problem with no SAT clauses and two constraints:

$$\begin{gathered} x_1 + \ldots + x_n \leq n/2 \\ x_1 + \ldots + x_n > n/2 \end{gathered}$$

Let us see how a (very) Lazy Approach would behave:

- Problem is obviously unsatisfiable
- Inconsistency explanations are of the form:

$$\begin{gathered} \neg x_{i_1} \vee \ldots \vee \neg x_{i_{n/2+1}} \\ x_{i_1} \vee \ldots \vee x_{i_{n/2}} \end{gathered}$$

Eager vs Lazy Approach (3)

Consider the problem with no SAT clauses and two constraints:

$$x_1 + \ldots + x_n \leq n/2$$
$$x_1 + \ldots + x_n > n/2$$

Let us see how a (very) Lazy Approach would behave:

- Problem is obviously unsatisfiable
- Inconsistency explanations are of the form:

$$\neg x_{i_1} \vee \ldots \vee \neg x_{i_{n/2+1}}$$
$$x_{i_1} \vee \ldots \vee x_{i_{n/2}}$$

- **All** $\binom{n}{\frac{n}{2}+1} + \binom{n}{n/2}$ explanations are needed to produce an unsatisfiable subset of clauses

Eager vs Lazy Approach (3)

Consider the problem with no SAT clauses and two constraints:

$$\begin{array}{c} x_1 + \ldots + x_n \leq n/2 \\ x_1 + \ldots + x_n > n/2 \end{array}$$

Let us see how a (very) Lazy Approach would behave:

- Problem is obviously unsatisfiable
- Inconsistency explanations are of the form:

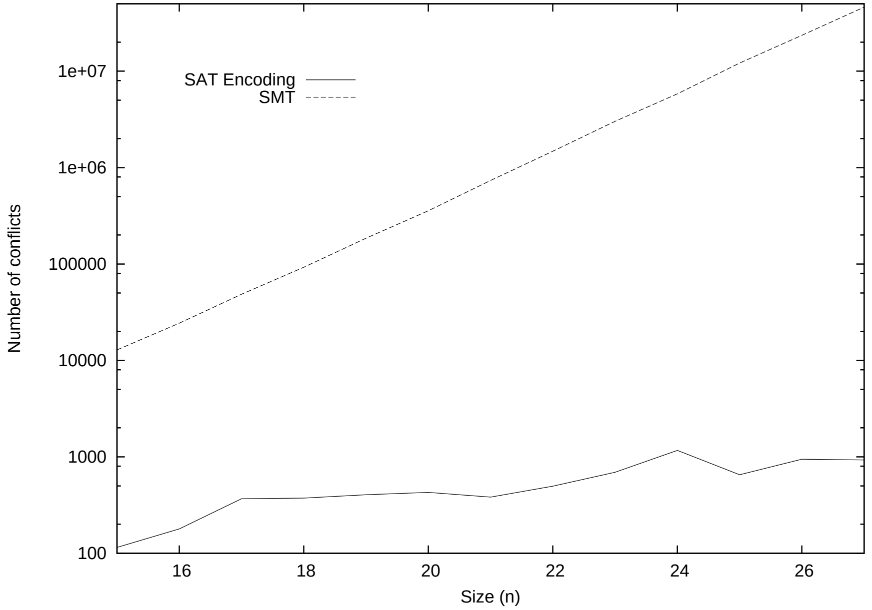
$$\begin{array}{c} \neg x_{i_1} \vee \ldots \vee \neg x_{i_{n/2+1}} \\ x_{i_1} \vee \ldots \vee x_{i_{n/2}} \end{array}$$

- **All** $\binom{n}{\frac{n}{2}+1} + \binom{n}{n/2}$ explanations are needed to produce an unsatisfiable subset of clauses
- Hence, runtime is exponential in *n*.

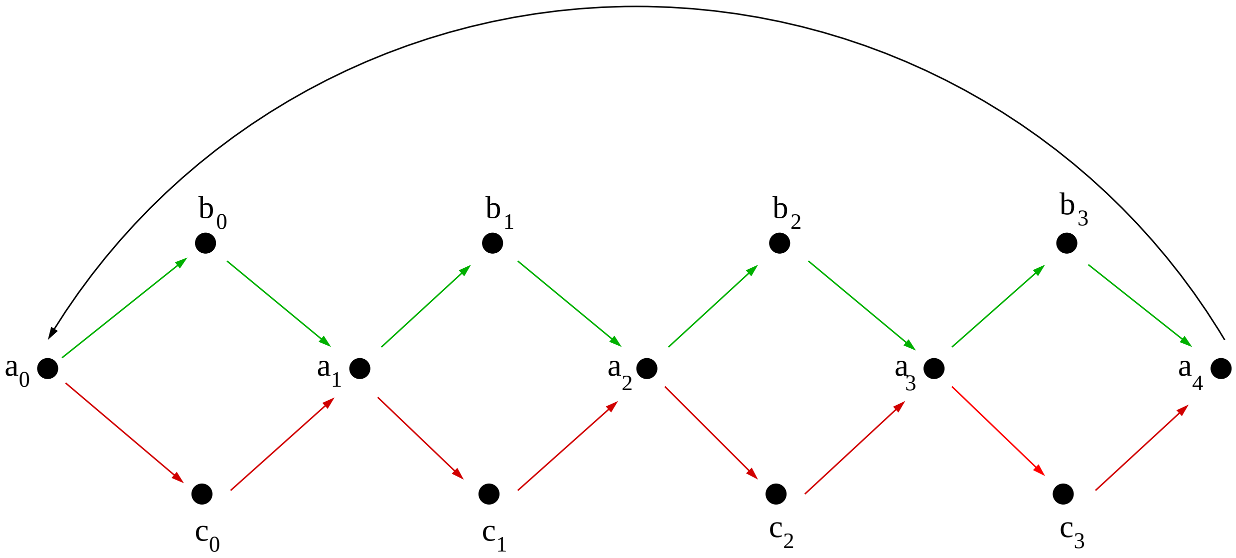
Eager vs Lazy approach (4)

What has happened?

- Lazy approach = lazily encoding (parts of) the theory into SAT
- Sometimes, only parts of the theory need to be encoded
- But in this example the whole constraint is encoded into SAT...
- ...and the encoding used is a very naive one
- Best here is a good SAT encoding with auxiliary variables



The diamonds example



$$a_n < a_o \wedge \bigwedge_{k=0}^{n-1} \left(\ (a_k < b_k \wedge b_k < a_{k+1}) \ \vee \ (a_k < c_k \wedge c_k < a_{k+1}) \ \right)$$

With these literals, only exponential refutations exist.

Introducing $a_0 < a_1, \ a_1 < a_2, \ldots$ allows linear refutations.

Other approaches

Previous examples show limitations of (DPLL(T))
There are more technical limitations out of the scope of this talk
Research on model-based procedures tries to address these issues:

- Linear Real Arithmetic
  - Generalizing DPLL to Richer Logics [MKS09]
  - Conflict Resolution [KTV09]
  - Natural Domain SMT [Cot10]
- Linear Integer Arithmetic
  - Cutting to the Chase [JdM13]
- Non-Linear Real Arithmetic
  - Solving Non-Linear Arithmetic [JM12]
- General Framework
  - Model-Constructing Satisfiability Calculus [JM13]
  - Satisfiability Modulo Theories and Assignments[BGS17]

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