Computing with SAT Oracles

Joao Marques-Silva, Alexey Ignatiev & Antonio Morgado

University of Lisbon, Portugal

August 12, 2019 | IJCAI Tutorial T20

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What is SAT?

- SAT is the decision problem for propositional logic
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- SAT is NP-complete [Coo71]

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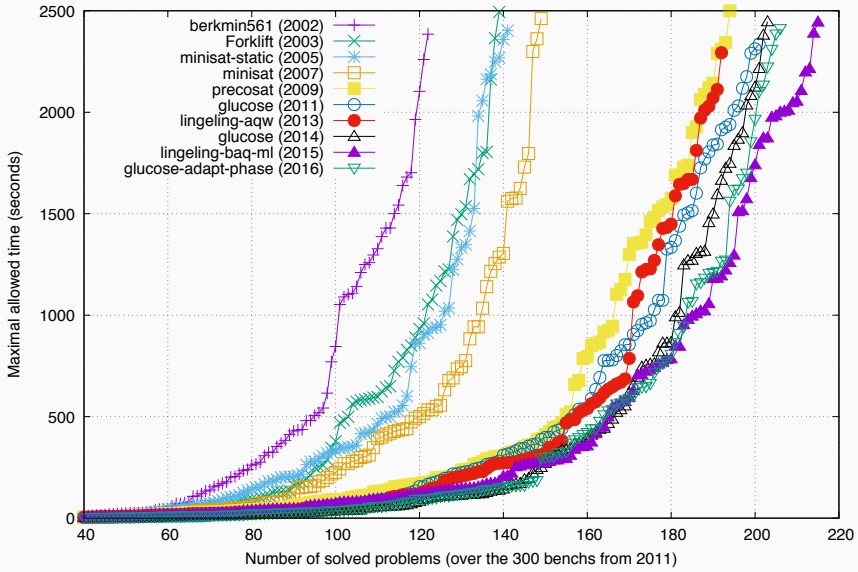
  - Conflict-Driven Clause Learning (CDCL)
  - (CDCL) SAT has impacted many different fields
  - Hundreds (thousands?) of practical applications

CDCL SAT solver (continued) improvement

[Source: Simon 2015]



How good are CDCL SAT solvers?

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- Sample SAT of solvers:
  1. POSIT: state of the art DPLL SAT solver in 1995
  2. GRASP: first CDCL SAT solver, state of the art 1995∼2000
  3. Minisat: CDCL SAT solver, state of the art until the late 00s
  4. Glucose: modern state of the art CDCL SAT solver
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- Example 1: model checking example (from IBM)
- Example 2: cooperative path finding (CPF)

How good are SAT solvers? – an example

- Cooperative pathfinding (CPF)
  - N agents on some grid/graph
  - Start positions
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  - Minimize makespan
  - Restricted planning problem

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- Concrete example
  - Gaming grid
  - 1039 vertices
  - 1928 edges
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```
*** tracker: a pathfinding tool ***

Initialization ... CPU Time: 0.004711
Number of variables: 113315
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Number of assumptions: 1
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Elapsed CPU Time: 0.830112
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Tentative makespan 24
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  - Formula w/ 2946190 variables!
- **Note:** In the early 90s, SAT solvers could solve formulas with a few hundred variables!

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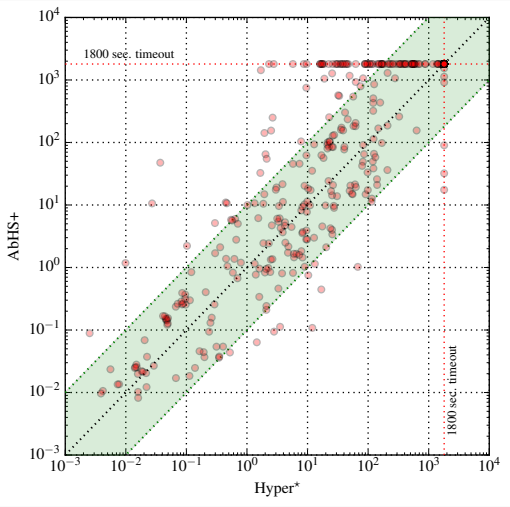
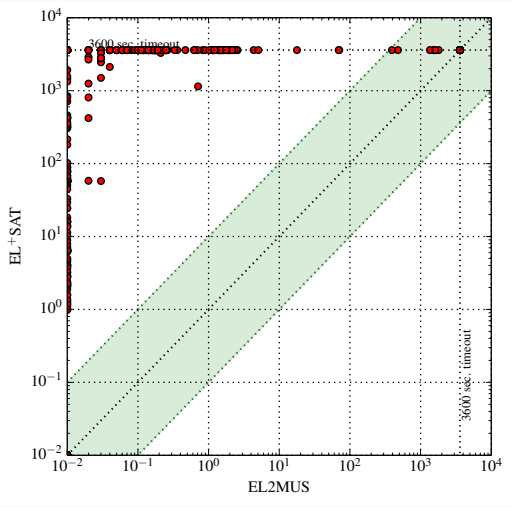
- Search space with 2832875 propositional variables (worst case):
  - # of assignments to $> 2.8 \times 10^6$ variables: $\gg 10^{840000}$ **!!**
  - **Obs:** SAT solvers at present (but formula dependent)

SAT can make the difference – propositional abduction



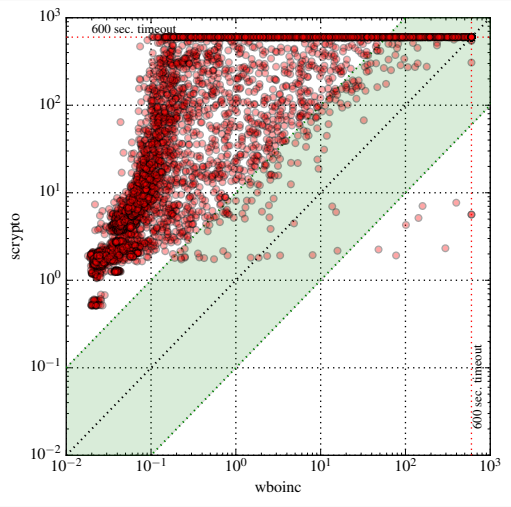
- Propositional abduction instances [IMM16]
  - Implicit hitting set dualization (IHSD)

SAT can make the difference – axiom pinpointing



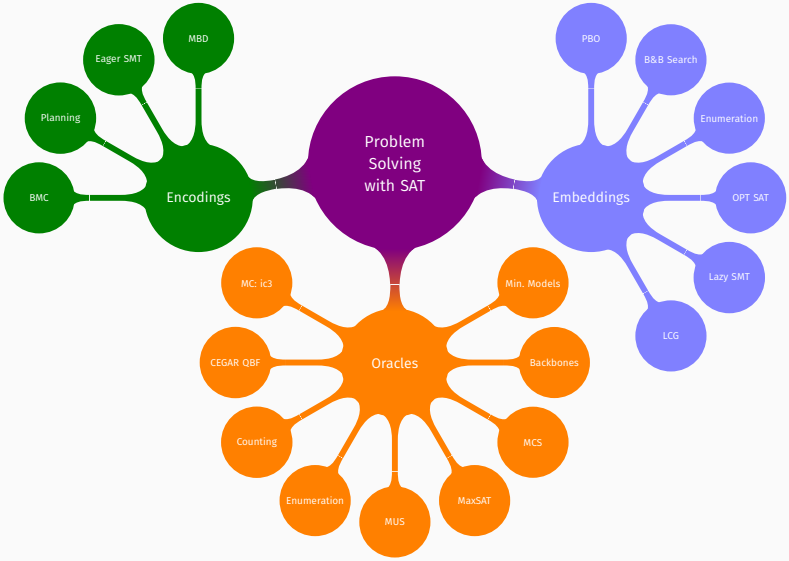
- $\mathcal{EL}^+$ medical ontologies [AMM15]
  - Minimal unsatisfiability (MUSes) & maximal satisfiability (MCSes) & Enumeration

SAT can make the difference – model based diagnosis



- Model-based diagnosis problem instances [MJIM15]
  - Maximum satisfiability (MaxSAT)

CDCL SAT is ubiquitous in problem solving



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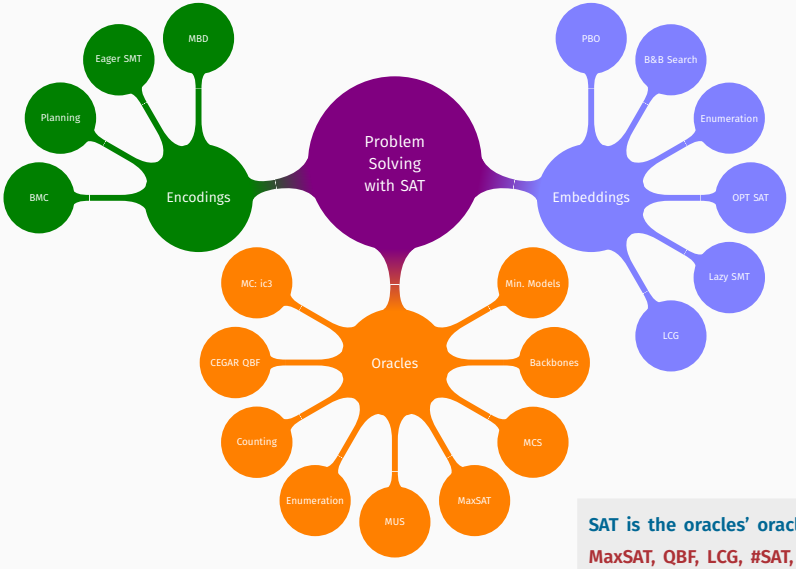

SAT is the oracles' oracle:
MaxSAT, QBF, LCG, #SAT, SMT, ASP, FOL, ...

Tutorial organization

- Part #1: Overview & basic definitions – Joao
- Part #2: Modern CDCL SAT solvers – Alexey
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1 Basic Definitions

Preliminaries

- Variables: $w, x, y, z, a, b, c, \ldots$
- Literals: $w, \bar{x}, \bar{y}, a, \ldots$, but also $\neg w, \neg y, \ldots$
- Clauses: disjunction of literals or set of literals
- Formula: conjunction of clauses or set of clauses
- Model (satisfying assignment): partial/total mapping from variables to $\{0, 1\}$ that satisfies formula
- Each clause can be satisfied, falsified, but also unit, unresolved
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$$\mathcal{F} \triangleq (r) \wedge (\bar{r} \vee s) \wedge (\bar{w} \vee a) \wedge (\bar{x} \vee b) \wedge (\bar{y} \vee \bar{z} \vee c) \wedge (\bar{b} \vee \bar{c} \vee d)$$

  - Example models:

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  - Example models:
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    - $\{r, s, \bar{x}, y, \bar{w}, z, \bar{a}, b, c, d\}$

Resolution

- Resolution rule: [DP60, Rob65]

$$\frac{(\alpha \vee x) \qquad (\beta \vee \bar{x})}{(\alpha \vee \beta)}$$

  - Complete proof system for propositional logic

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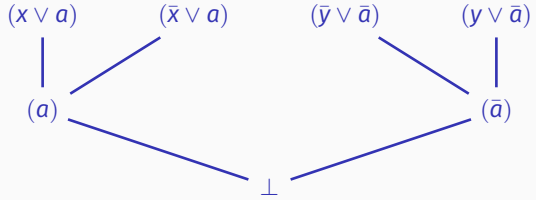

  - Extensively used with (CDCL) SAT solvers

Unit propagation

$$\begin{aligned}\mathcal{F} \quad = \quad & (r) \wedge (\bar{r} \vee s) \wedge \\ & (\bar{w} \vee a) \wedge (\bar{x} \vee \bar{a} \vee b) \wedge \\ & (\bar{y} \vee \bar{z} \vee c) \wedge (\bar{b} \vee \bar{c} \vee d)\end{aligned}$$

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- Decisions / Variable Branchings:
  $w = 1, x = 1, y = 1, z = 1$

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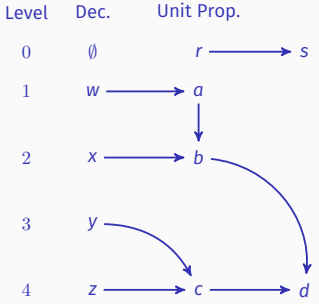

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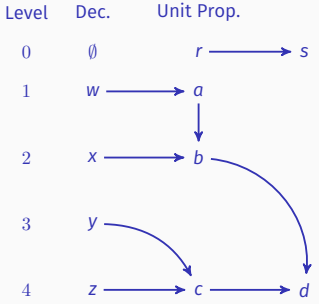

- Decisions / Variable Branchings:
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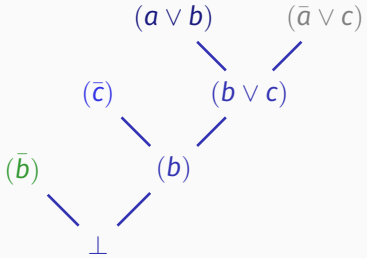
- **Unit clause rule**: if clause is unit, its sole literal **must** be satisfied

- Additional definitions:
  - Antecedent (or reason) of an implied assignment
    - $(\bar{b} \vee \bar{c} \vee d)$ for d
  - Associate assignment with decision levels
    - $w = 1 \,@\, 1, x = 1 \,@\, 2, y = 1 \,@\, 3, z = 1 \,@\, 4$
    - $r = 1 \,@\, 0, d = 1 \,@\, 4$, ...

Resolution proofs

- Refutation of unsatisfiable formula by iterated resolution operations yields resolution proof
- An example:
  $\mathcal{F} = (\bar{c}) \wedge (\bar{b}) \wedge (\bar{a} \vee c) \wedge (a \vee b) \wedge (a \vee \bar{d}) \wedge (\bar{a} \vee \bar{d})$
- Resolution proof:



- Modern SAT solvers can generate resolution proofs from learned clauses [ZM03]

Unsatisfiable cores & proof traces

- CNF formula:

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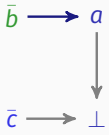
Level	Dec.	Unit Prop.
0	$\emptyset$	$\bar{b} \rightarrow a$
		$\bar{c} \rightarrow \bot$



Implication graph with conflict

Unsatisfiable cores & proof traces

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Proof trace $\bot$: $(\bar{a} \vee c)$ $(a \vee b)$ $(\bar{c})$ $(\bar{b})$

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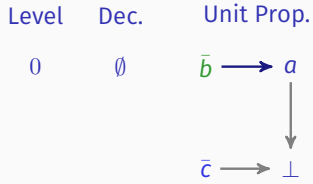

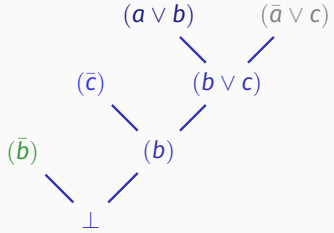

Resolution proof follows structure of conflicts

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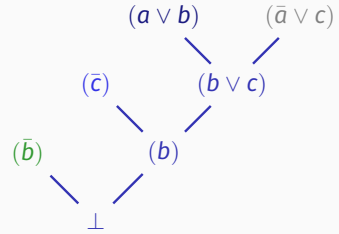
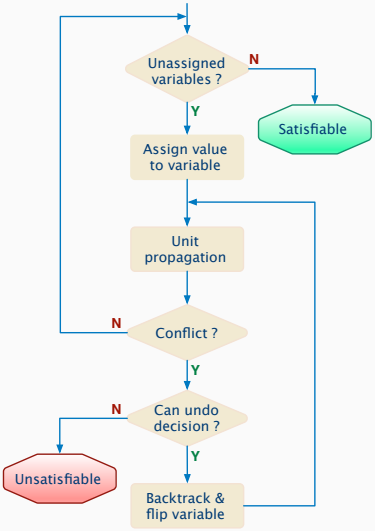


Unsatisfiable subformula (core): $(\bar{c}), (\bar{b}), (\bar{a} \vee c), (a \vee b)$

The DPLL algorithm

[DP60, DLL62]



- Optional: pure literal rule

The DPLL algorithm

[DP60, DLL62]

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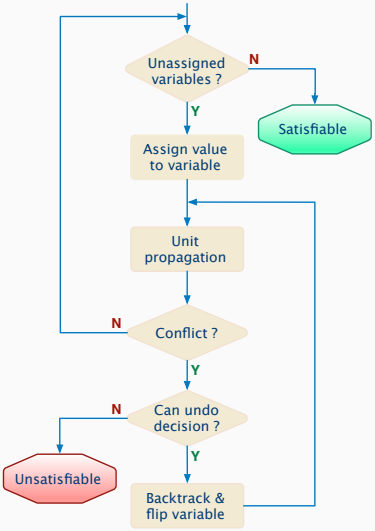

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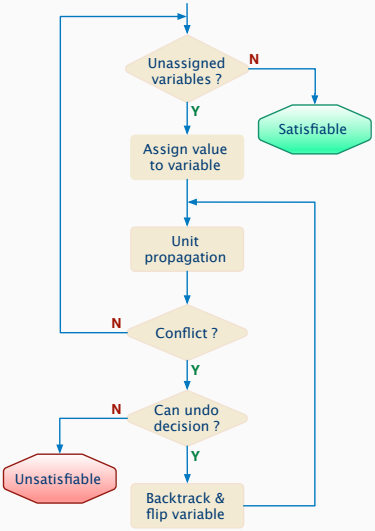

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Level	Dec.	Unit Prop.
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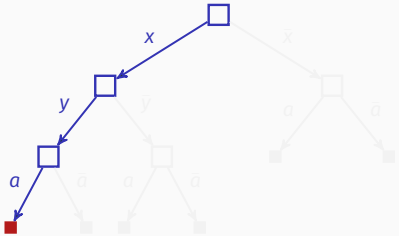

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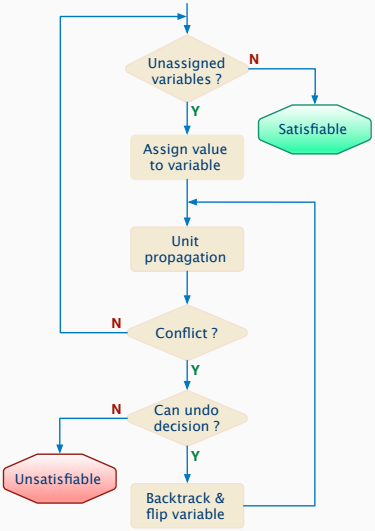

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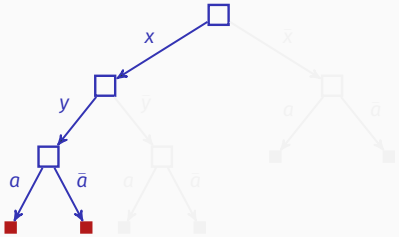

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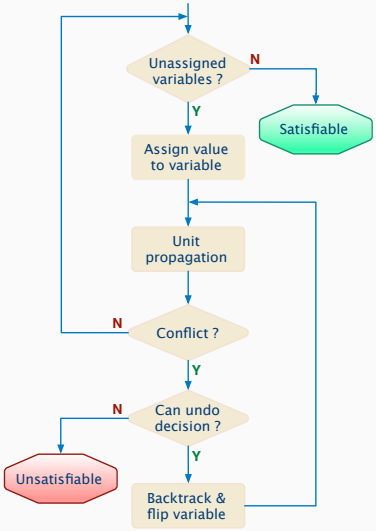

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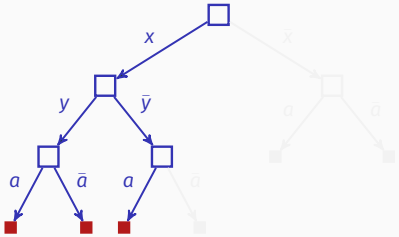

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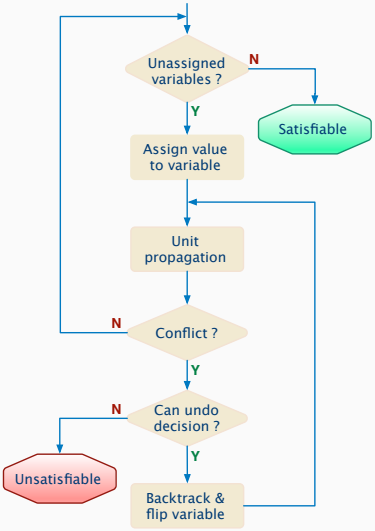

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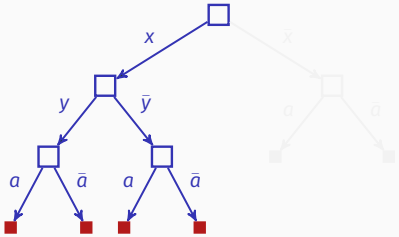

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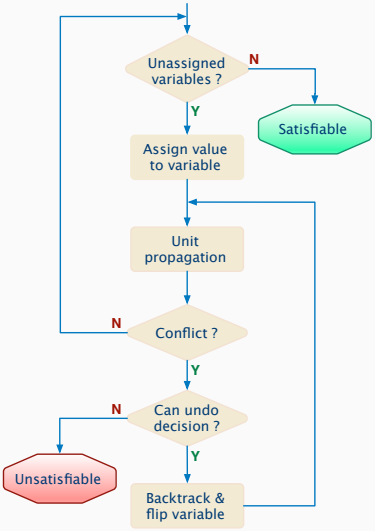

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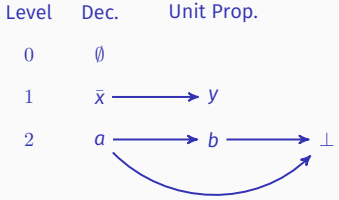

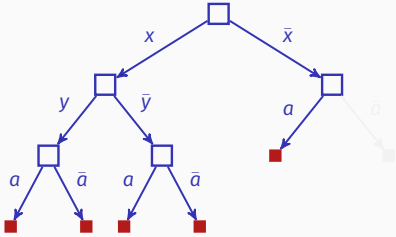

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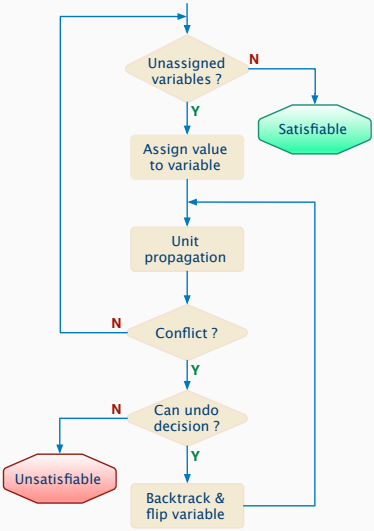

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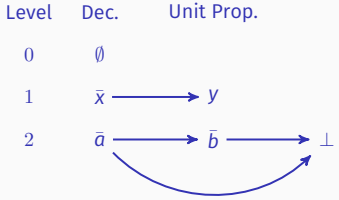

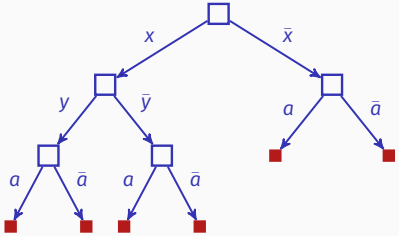

2 CDCL SAT Solvers

What is a CDCL SAT solver?

- Extend DPLL SAT solver with: [DP60, DLL62]
  - Clause learning & non-chronological backtracking [MS95, MSS96b, MSS99]
  - Search restarts [GSC97, BMS00, Hua07, Bie08, LOM+18]
  - Lazy data structures
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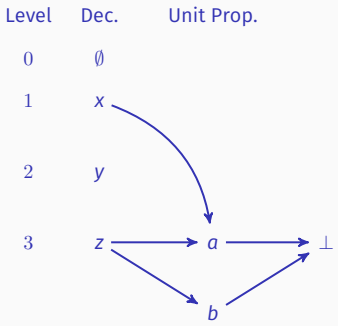
- Extend DPLL SAT solver with: [DP60, DLL62]
  - Clause learning & non-chronological backtracking [MS95, MSS96b, MSS99]
    - Exploit UIPs [MS95, MSS99, ZMMM01, SSS12]
    - Minimize learned clauses [SB09, Gel09, LLX+17]
    - Opportunistically delete clauses [MSS96b, MSS99, GN02, AS09]
  - Search restarts [GSC97, BMS00, Hua07, Bie08, LOM+18]
  - Lazy data structures
    - Watched literals [MMZ+01]
  - Conflict-guided branching
    - Lightweight branching heuristics [MMZ+01]
    - Phase saving [PD07]
  - ...

Clause Learning, UIPs & Minimization

Clause learning



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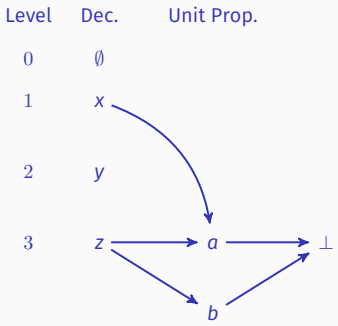
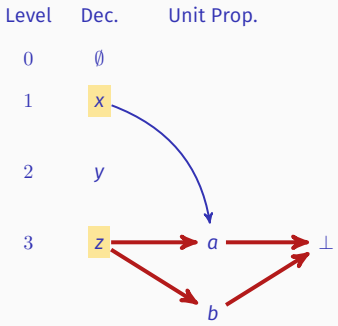


- Analyze conflict

[MS95, MSS96a, MSS96a, MSS96b, MSS99]

Clause learning



- Analyze conflict [MS95, MSS96a, MSS96a, MSS96b, MSS99]
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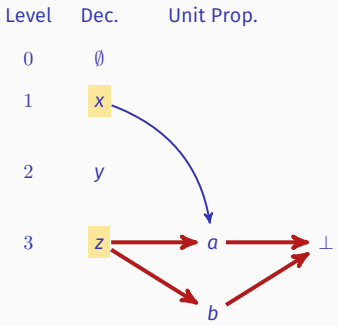

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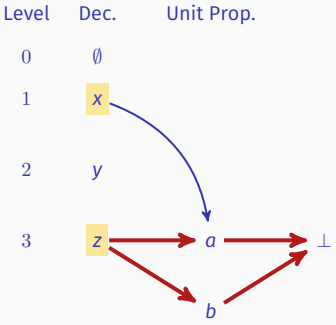

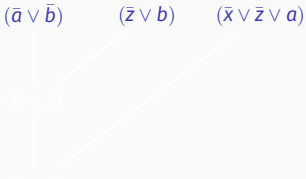

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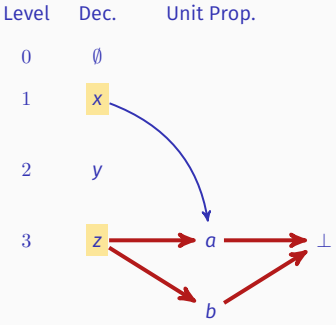

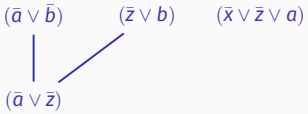

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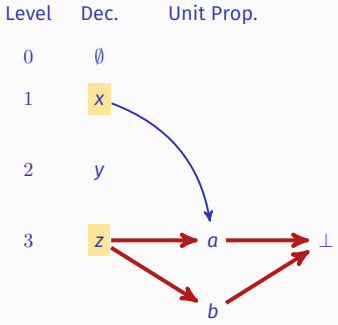

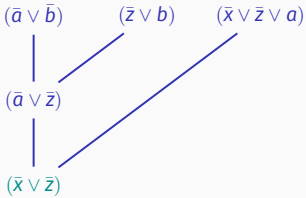

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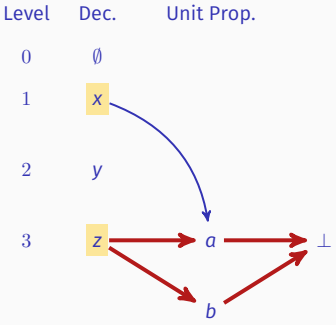

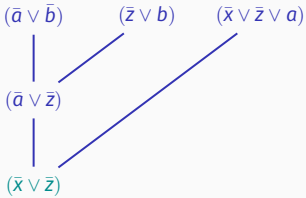

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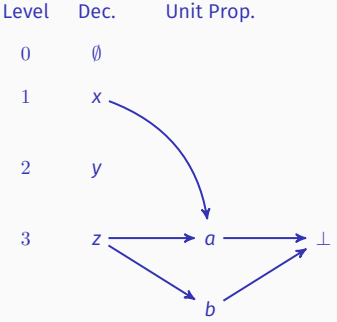
  - Learned clauses result from (**selected**) resolution operations

Clause learning – after backtracking



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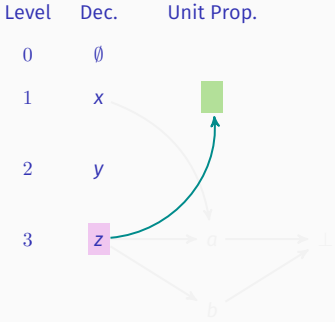

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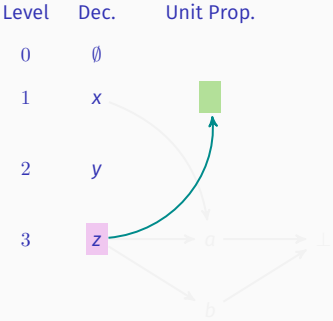

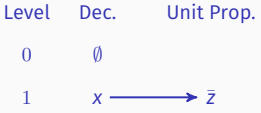

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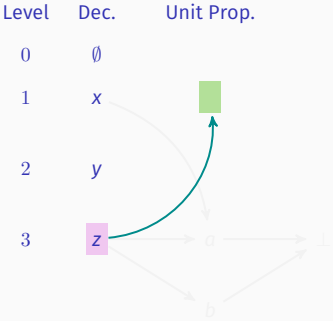
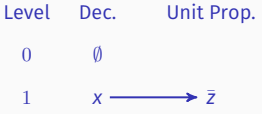
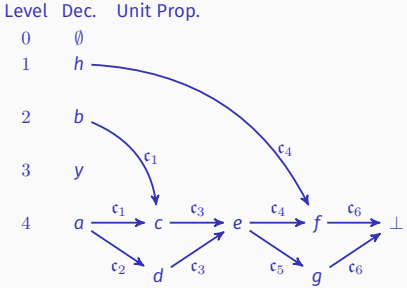




- Clause $(\bar{x} \vee \bar{z})$ is asserting at decision level 1
- Learned clauses are asserting (with exceptions) [MS95, MSS96b, MSS99]
- Backtracking differs from plain DPLL:
  - Always bactrack after a conflict [MMZ+01]

Quiz – conflict analysis



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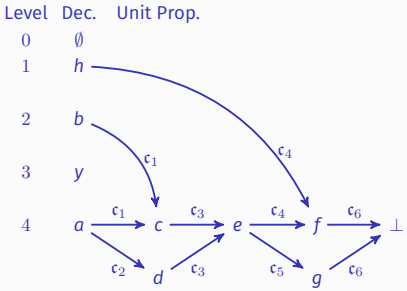

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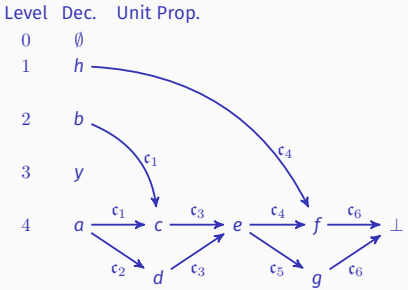

Step	Var Queue	Extract Var	Antecedent	Recorded Lits	Added to Queue
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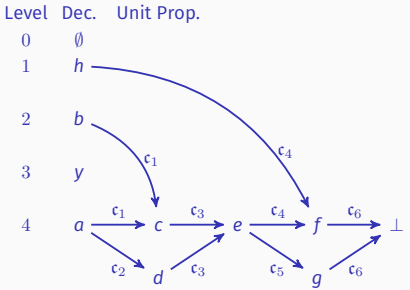
Quiz – conflict analysis



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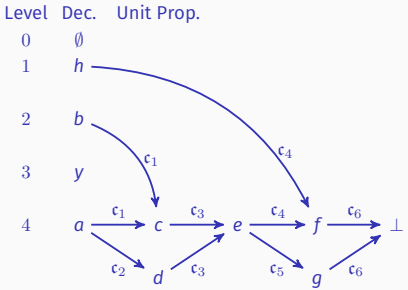
Quiz – conflict analysis



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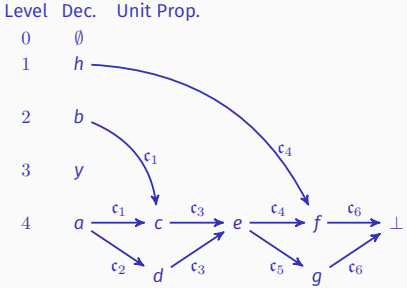
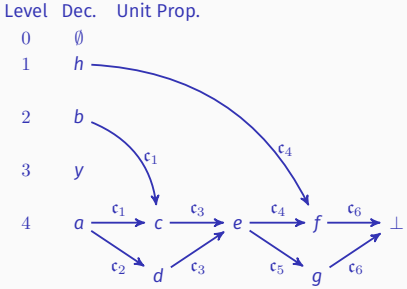
1	$[f, g]$	f	c_4	$\{\bar{h}\}$	$\{e\}$
2	$[g, e]$	g	c_5	$\{\bar{h}\}$	$\emptyset$
3	$[e]$	e	c_3	$\{\bar{h}\}$	$\{c, d\}$

Quiz – conflict analysis



Step	Var Queue	Extract Var	Antecedent	Recorded Lits	Added to Queue
0	–	$\bot$	c_6	$\emptyset$	$\{f, g\}$
1	$[f, g]$	f	c_4	$\{\bar{h}\}$	$\{e\}$
2	$[g, e]$	g	c_5	$\{\bar{h}\}$	$\emptyset$
3	$[e]$	e	c_3	$\{\bar{h}\}$	$\{c, d\}$
4	$[c, d]$	c	c_1	$\{\bar{h}, \bar{b}\}$	$\{a\}$

Quiz – conflict analysis



Step	Var Queue	Extract Var	Antecedent	Recorded Lits	Added to Queue
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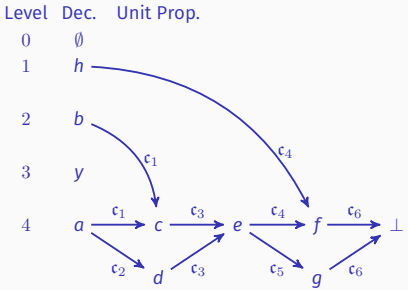
3	$[e]$	e	c_3	$\{\bar{h}\}$	$\{c, d\}$
4	$[c, d]$	c	c_1	$\{\bar{h}, \bar{b}\}$	$\{a\}$
5	$[d, a]$	d	c_2	$\{\bar{h}, \bar{b}\}$	$\emptyset$

Quiz – conflict analysis



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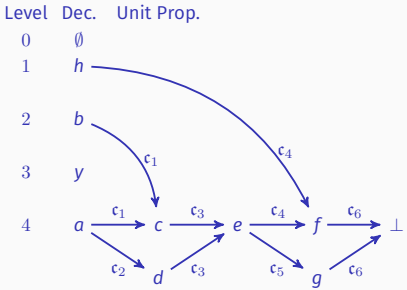
6	$[a]$	a	dec var	$\{\bar{h}, \bar{b}, \bar{a}\}$	–

Quiz – conflict analysis



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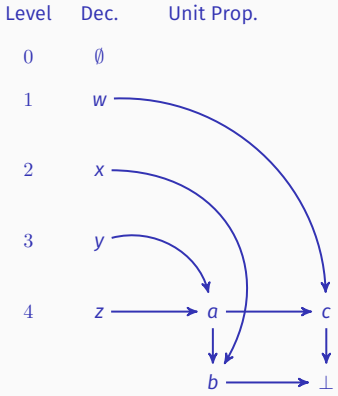
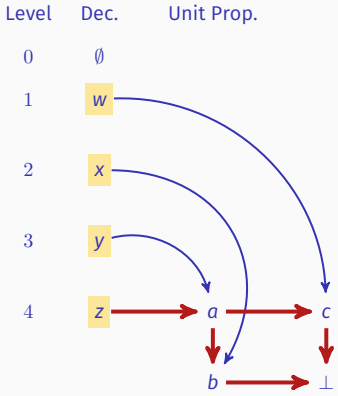
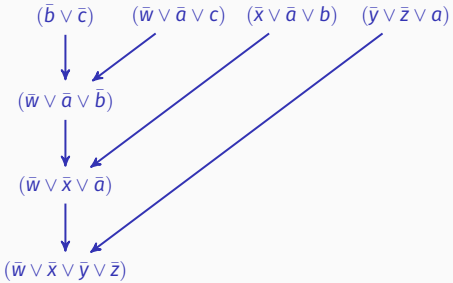
7	$[]$	–	–	$\{\bar{h}, \bar{b}, \bar{a}\}$	–

Unique implication points (UIPs)



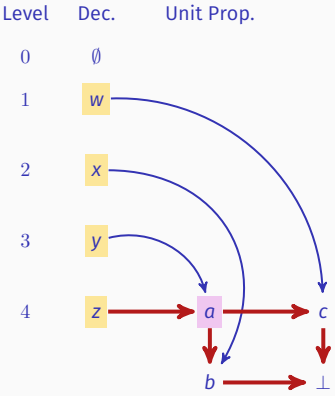
Unique implication points (UIPs)

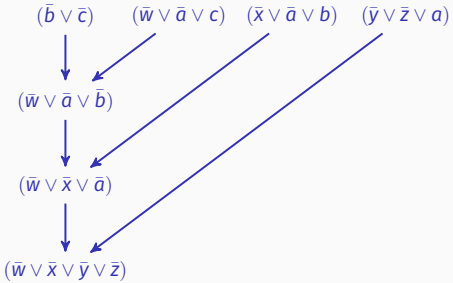




- Learn clause $(\bar{w} \vee \bar{x} \vee \bar{y} \vee \bar{z})$

Unique implication points (UIPs)

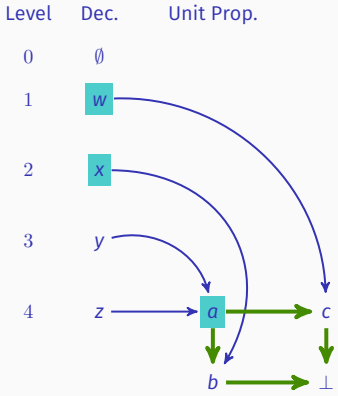


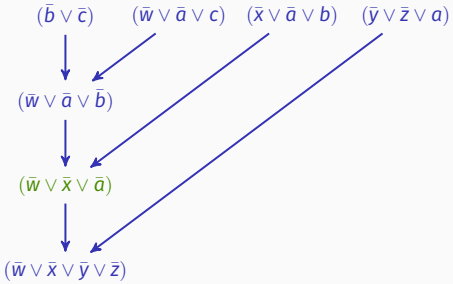


- Learn clause $(\bar{w} \vee \bar{x} \vee \bar{y} \vee \bar{z})$
- But a is an UIP
  - Dominator in DAG for decision level 4

[MS95, MSS99]

Unique implication points (UIPs)





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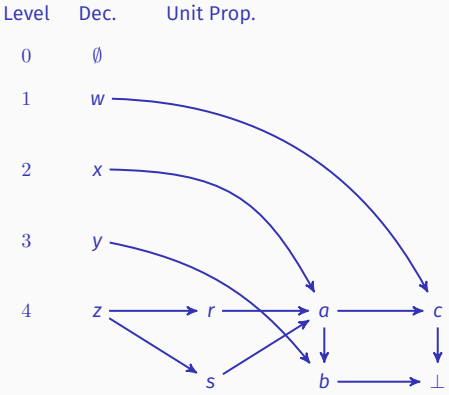
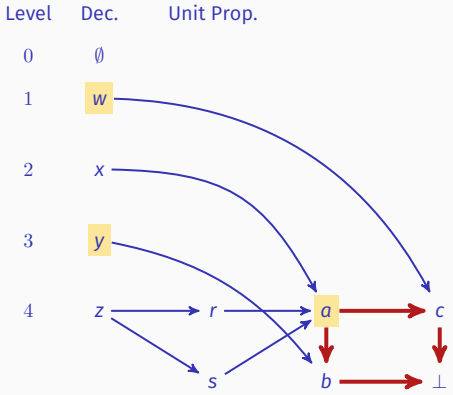
[MS95, MSS99]

Multiple UIPs



Multiple UIPs



- First UIP:
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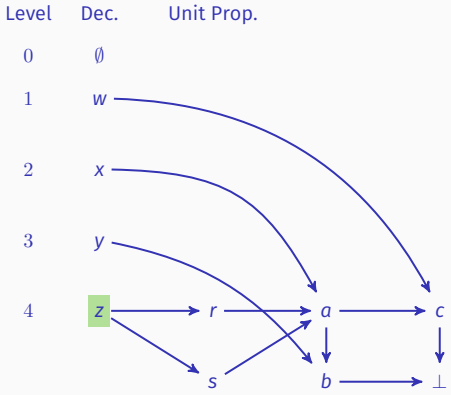
Multiple UIPs



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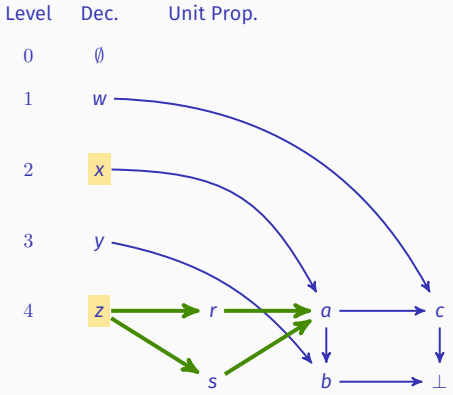
Multiple UIPs



- First UIP:
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- But there can be more than 1 UIP
- Second UIP:
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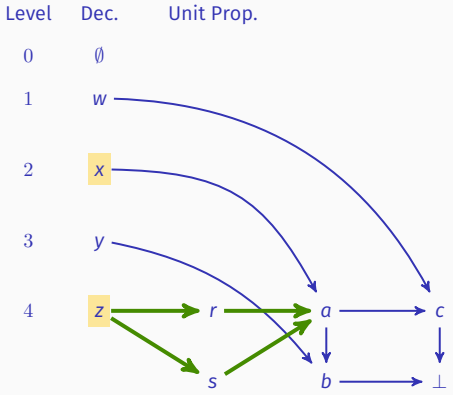
  - Clause is not asserting

Multiple UIPs



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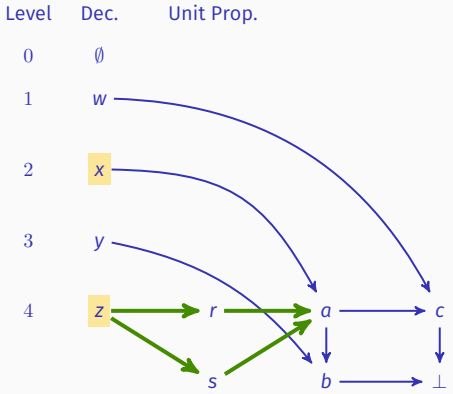
  - Compare with $(\overline{w} \vee \overline{x} \vee \overline{y} \vee \overline{z})$

Multiple UIPs



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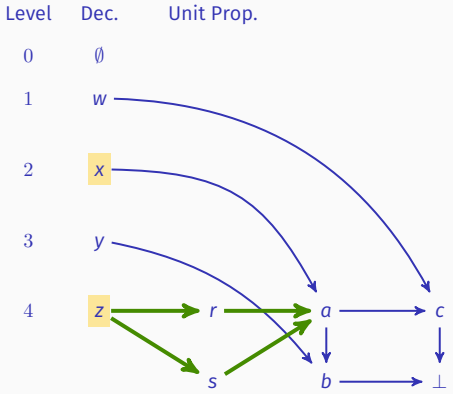
- Multiple UIPs proposed in GRASP [MS95, MSS99]
  - First UIP learning proposed in Chaff [MMZ+01]
- Not used in recent state of the art CDCL SAT solvers

Multiple UIPs



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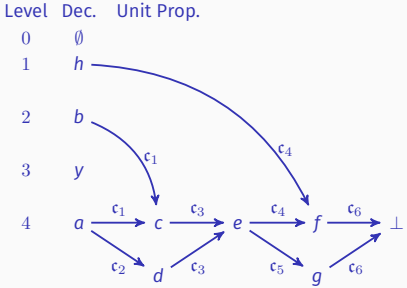
- Multiple UIPs proposed in GRASP [MS95, MSS99]
  - First UIP learning proposed in Chaff [MMZ+01]
- Not used in recent state of the art CDCL SAT solvers
- Recent results show it can be beneficial on some instances [SSS12]

Quiz – conflict analysis with UIP(s)



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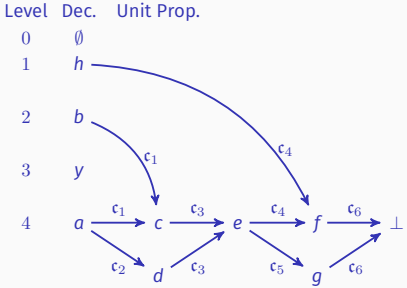

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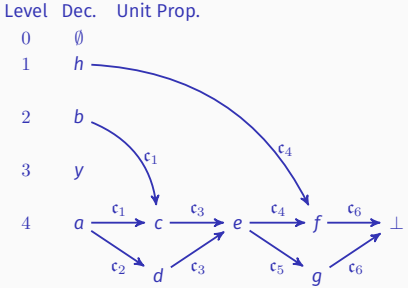

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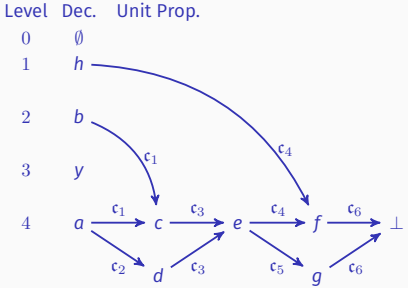

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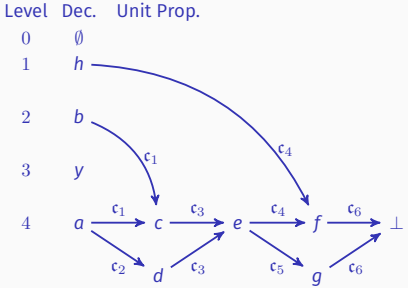

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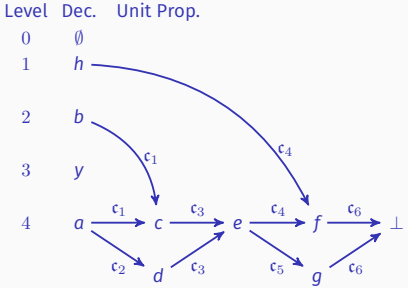

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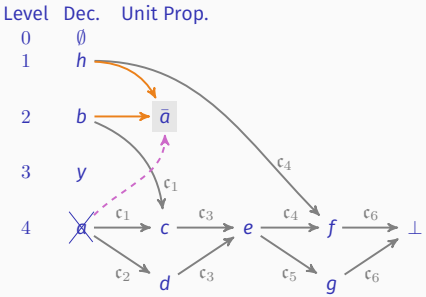
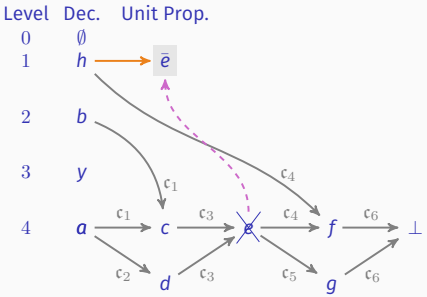
Quiz (Cont.) – non-chronological backtracking

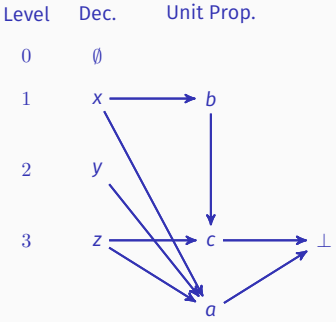
Without UIP:



With UIP:



Clause minimization I



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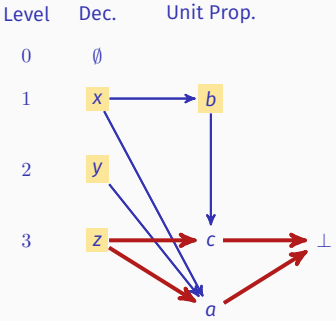
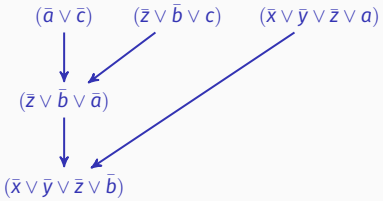
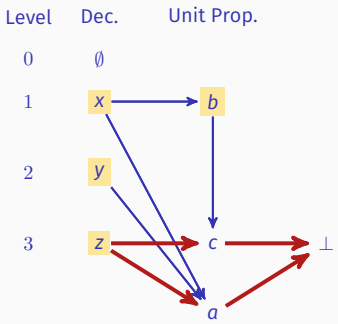
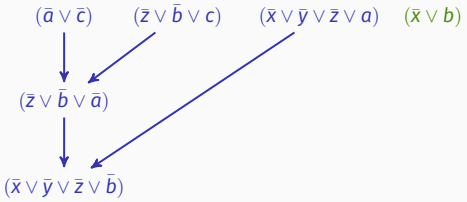




- Learn clause $(\bar{x} \vee \bar{y} \vee \bar{z} \vee \bar{b})$

Clause minimization I

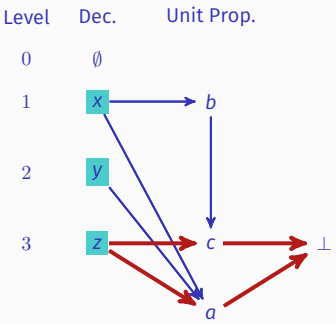


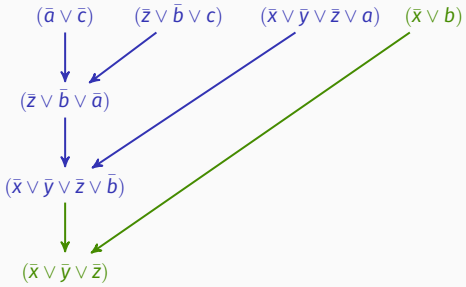


- Learn clause $(\bar{x} \vee \bar{y} \vee \bar{z} \vee \bar{b})$
- Apply self-subsuming resolution (i.e. local minimization)

[SB09]

Clause minimization I





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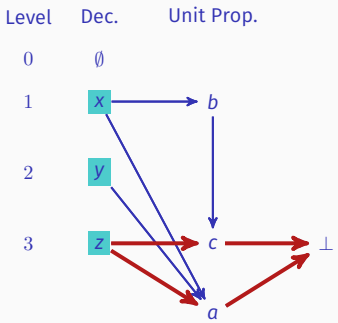
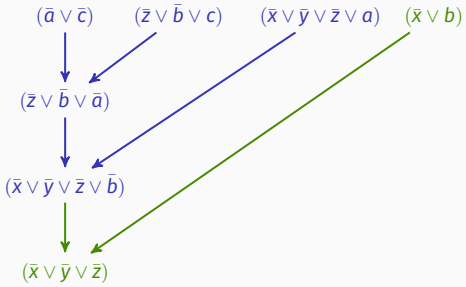
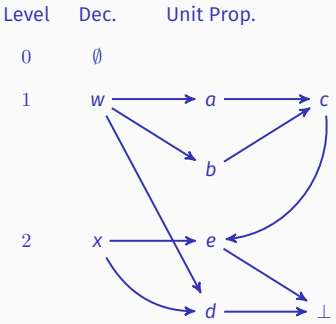
Clause minimization I



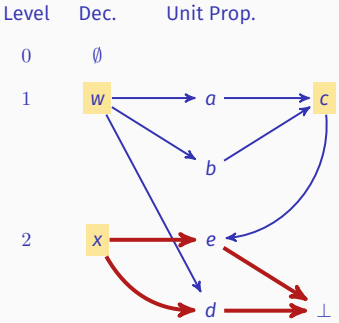


- ~~Learn clause $(\bar{x} \vee \bar{y} \vee \bar{z} \vee \bar{b})$~~
- Apply self-subsuming resolution (i.e. local minimization) [SB09]
- Learn clause $(\bar{x} \vee \bar{y} \vee \bar{z})$

Clause minimization II

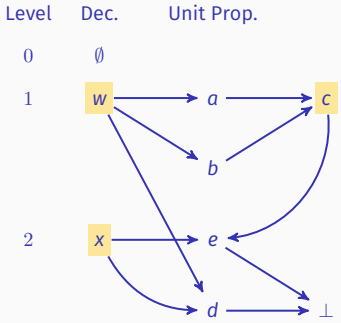


Clause minimization II



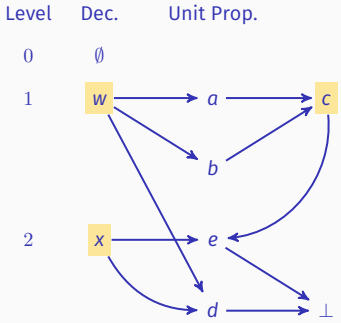
- Learn clause $(\overline{w} \vee \bar{x} \vee \bar{c})$

Clause minimization II



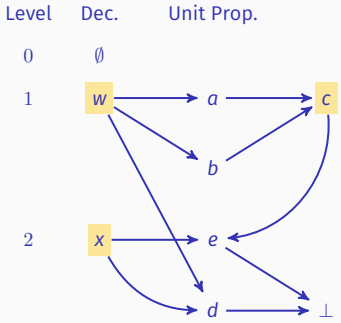
- Learn clause $(\bar{w} \vee \bar{x} \vee \bar{c})$
- Cannot apply self-subsuming resolution
  - Resolving with reason of c yields $(\bar{w} \vee \bar{x} \vee \bar{a} \vee \bar{b})$

Clause minimization II



- Learn clause $(\bar{w} \vee \bar{x} \vee \bar{c})$
- Cannot apply self-subsuming resolution
  - Resolving with reason of c yields $(\bar{w} \vee \bar{x} \vee \bar{a} \vee \bar{b})$
- Can apply recursive minimization

Clause minimization II

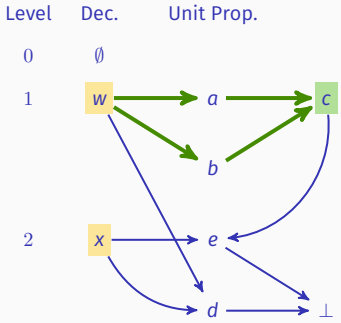


- ~~Learn clause $(\bar{w} \vee \bar{x} \vee \bar{c})$~~
- Cannot apply self-subsuming resolution
  - Resolving with reason of c yields $(\bar{w} \vee \bar{x} \vee \bar{a} \vee \bar{b})$
- Can apply recursive minimization

- Marked nodes: literals in learned clause

[SB09]

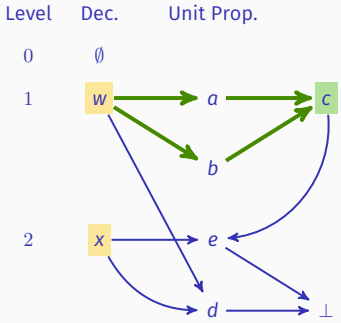
Clause minimization II



- ~~Learn clause $(\bar{w} \vee \bar{x} \vee \bar{c})$~~
- Cannot apply self-subsuming resolution
  - Resolving with reason of c yields $(\bar{w} \vee \bar{x} \vee \bar{a} \vee \bar{b})$
- Can apply recursive minimization

- Marked nodes: literals in learned clause [SB09]
- Trace back from c until marked nodes or new decision nodes
  - Drop literal c if only marked nodes visited

Clause minimization II



- ~~Learn clause $(\bar{w} \vee \bar{x} \vee \bar{c})$~~
- Cannot apply self-subsuming resolution
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- Learn clause $(\bar{w} \vee \bar{x})$

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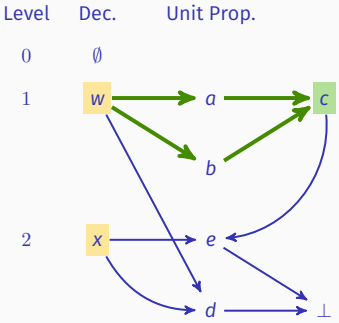
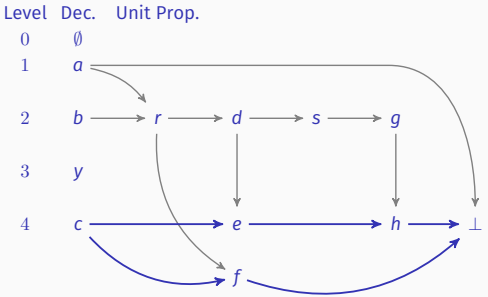
Clause minimization II



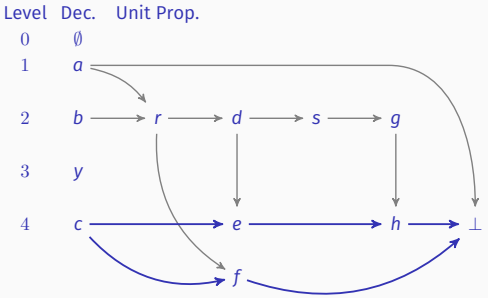
- ~~Learn clause $(\bar{w} \vee \bar{x} \vee \bar{c})$~~
- Cannot apply self-subsuming resolution
  - Resolving with reason of c yields $(\bar{w} \vee \bar{x} \vee \bar{a} \vee \bar{b})$
- Can apply recursive minimization
- Learn clause $(\bar{w} \vee \bar{x})$

- Marked nodes: literals in learned clause [SB09]
- Trace back from c until marked nodes or new decision nodes
  - Drop literal c if only marked nodes visited
- Recursive minimization runs in (amortized) linear time

Quiz – conflict clause minimization



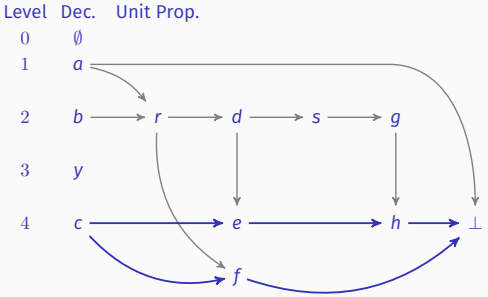
Quiz – conflict clause minimization



Learned clause: $(\bar{a} \vee \bar{r} \vee \bar{c} \vee \bar{d} \vee \bar{g})$

Minimized clause: $(\bar{a} \vee \bar{r} \vee \bar{c} \vee \bar{d} \vee \bar{g})$

Quiz – conflict clause minimization

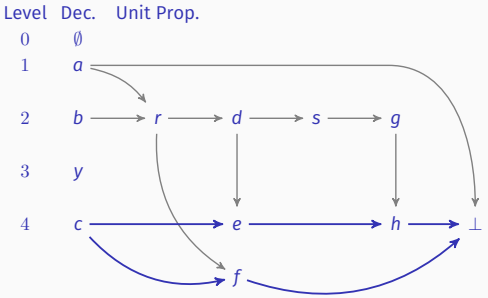


Learned clause: $(\bar{a} \vee \bar{r} \vee \bar{c} \vee \bar{d} \vee \bar{g})$

Minimized clause: $(\bar{a} \vee \bar{r} \vee \bar{c} \vee \bar{d} \vee \bar{g})$

Target	Curr Var	Marked	Unmarked	Vars to Trace	Action

Quiz – conflict clause minimization



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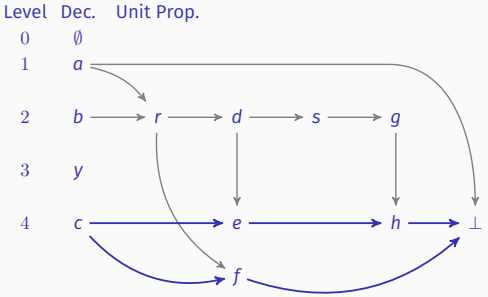
Minimized clause: $(\bar{a} \vee \bar{r} \vee \bar{c} \vee \bar{d} \vee \bar{g})$

Target	Curr Var	Marked	Unmarked	Vars to Trace	Action
g	g	$\{a, d, r, c\}$	$\emptyset$	$[s]$	–

Quiz – conflict clause minimization



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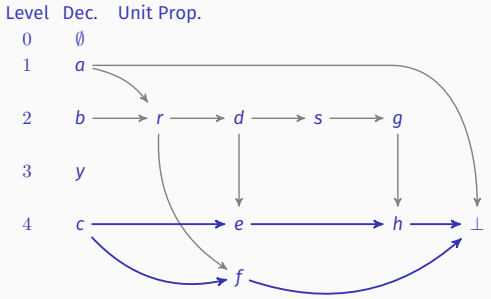
Minimized clause: $(\bar{a} \vee \bar{r} \vee \bar{c} \vee \bar{d} \vee \bar{g})$

Target	Curr Var	Marked	Unmarked	Vars to Trace	Action
g	g	$\{a, d, r, c\}$	$\emptyset$	$[s]$	–
g	s	$\{a, d, r, c\}$	$\emptyset$	$[d]$	–

Quiz – conflict clause minimization

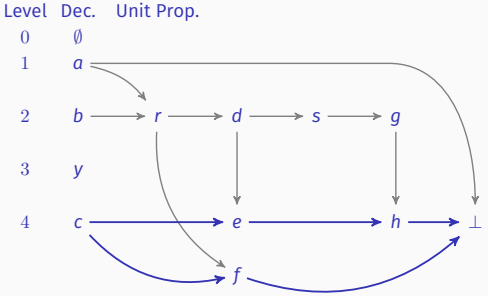


Learned clause: $(\bar{a} \vee \bar{r} \vee \bar{c} \vee \bar{d} \vee \bar{g})$

Minimized clause: $(\bar{a} \vee \bar{r} \vee \bar{c} \vee \bar{d} \vee \bar{g})$

Target	Curr Var	Marked	Unmarked	Vars to Trace	Action
g	g	$\{a, d, r, c\}$	$\emptyset$	$[s]$	–
g	s	$\{a, d, r, c\}$	$\emptyset$	$[d]$	–
g	d	$\{a, d, r, c\}$	$\emptyset$	$[]$	d marked, skip

Quiz – conflict clause minimization

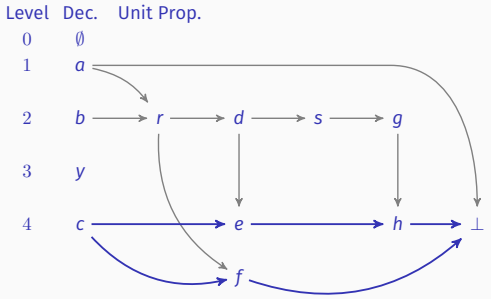


Learned clause: $(\bar{a} \vee \bar{r} \vee \bar{c} \vee \bar{d} \vee \bar{g})$

Minimized clause: $(\bar{a} \vee \bar{r} \vee \bar{c} \vee \bar{d} \quad)$

Target	Curr Var	Marked	Unmarked	Vars to Trace	Action
g	g	$\{a, d, r, c\}$	$\emptyset$	$[s]$	–
g	s	$\{a, d, r, c\}$	$\emptyset$	$[d]$	–
g	d	$\{a, d, r, c\}$	$\emptyset$	$[]$	d marked, skip
g	–	$\{a, d, r, c\}$	$\emptyset$	$[]$	no unmarked vars; $\therefore$ drop g

Quiz – conflict clause minimization

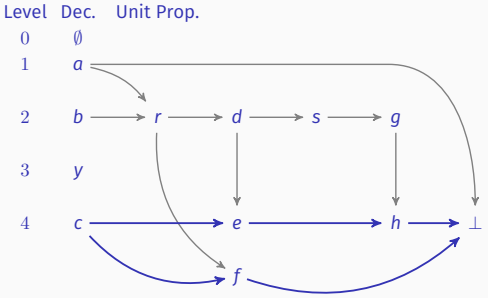


Learned clause: $(\bar{a} \vee \bar{r} \vee \bar{c} \vee \bar{d} \vee \bar{g})$

Minimized clause: $(\bar{a} \vee \bar{r} \vee \bar{c} \vee \bar{d} \quad)$

Target	Curr Var	Marked	Unmarked	Vars to Trace	Action
g	g	$\{a, d, r, c\}$	$\emptyset$	$[s]$	–
g	s	$\{a, d, r, c\}$	$\emptyset$	$[d]$	–
g	d	$\{a, d, r, c\}$	$\emptyset$	$[]$	d marked, skip
g	–	$\{a, d, r, c\}$	$\emptyset$	$[]$	no unmarked vars; $\therefore$ drop g
d	d	$\{a, r, c\}$	$\emptyset$	$[r]$	–

Quiz – conflict clause minimization

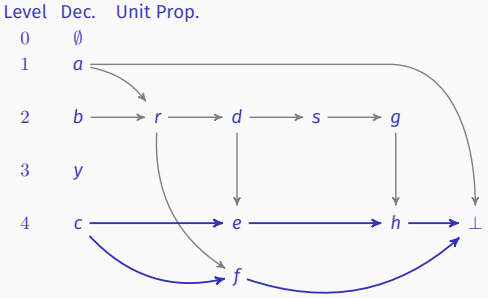


Learned clause: $(\bar{a} \vee \bar{r} \vee \bar{c} \vee \bar{d} \vee \bar{g})$

Minimized clause: $(\bar{a} \vee \bar{r} \vee \bar{c} \vee \bar{d} \quad)$

Target	Curr Var	Marked	Unmarked	Vars to Trace	Action
g	g	$\{a, d, r, c\}$	$\emptyset$	$[s]$	–
g	s	$\{a, d, r, c\}$	$\emptyset$	$[d]$	–
g	d	$\{a, d, r, c\}$	$\emptyset$	$[]$	d marked, skip
g	–	$\{a, d, r, c\}$	$\emptyset$	$[]$	no unmarked vars; $\therefore$ drop g
d	d	$\{a, r, c\}$	$\emptyset$	$[r]$	–
d	r	$\{a, r, c\}$	$\emptyset$	$[]$	r marked, skip

Quiz – conflict clause minimization

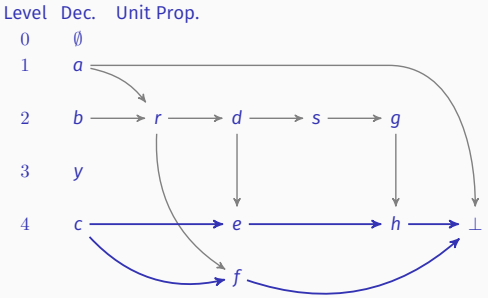


Learned clause: $(\bar{a} \vee \bar{r} \vee \bar{c} \vee \bar{d} \vee \bar{g})$

Minimized clause: $(\bar{a} \vee \bar{r} \vee \bar{c} \quad)$

Target	Curr Var	Marked	Unmarked	Vars to Trace	Action
g	g	$\{a, d, r, c\}$	$\emptyset$	$[s]$	–
g	s	$\{a, d, r, c\}$	$\emptyset$	$[d]$	–
g	d	$\{a, d, r, c\}$	$\emptyset$	$[]$	d marked, skip
g	–	$\{a, d, r, c\}$	$\emptyset$	$[]$	no unmarked vars; $\therefore$ drop g
d	d	$\{a, r, c\}$	$\emptyset$	$[r]$	–
d	r	$\{a, r, c\}$	$\emptyset$	$[]$	r marked, skip
d	–	$\{a, r, c\}$	$\emptyset$	$[]$	no unmarked vars; $\therefore$ drop d

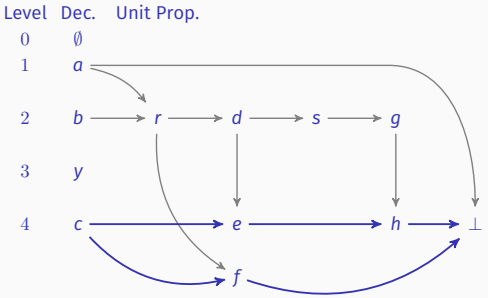
Quiz – conflict clause minimization (cont.)



Learned clause: $(\bar{a} \vee \bar{r} \vee \bar{c} \vee \bar{d} \vee \bar{g})$

Minimized clause: $(\bar{a} \vee \bar{r} \vee \bar{c})$

Quiz – conflict clause minimization (cont.)

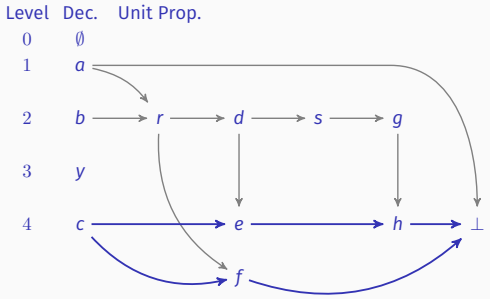


Learned clause: $(\bar{a} \vee \bar{r} \vee \bar{c} \vee \bar{d} \vee \bar{g})$

Minimized clause: $(\bar{a} \vee \bar{r} \vee \bar{c})$

Target	Curr Var	Marked	Unmarked	Vars to Trace	Action

Quiz – conflict clause minimization (cont.)

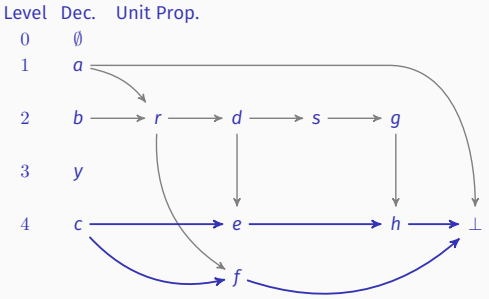


Learned clause: $(\bar{a} \vee \bar{r} \vee \bar{c} \vee \bar{d} \vee \bar{g})$

Minimized clause: $(\bar{a} \vee \bar{r} \vee \bar{c})$

Target	Curr Var	Marked	Unmarked	Vars to Trace	Action
r	r	$\{a, c\}$	$\emptyset$	$[a, b]$	–
r	a	$\{a, c\}$	$\emptyset$	$[b]$	a marked
r	b	$\{a, c\}$	$\{b\}$	$[]$	b decision & unmarked
r	–	$\{a, c\}$	$\{b\}$	$[]$	unmarked vars; $\therefore$ keep r

Quiz – conflict clause minimization (cont.)



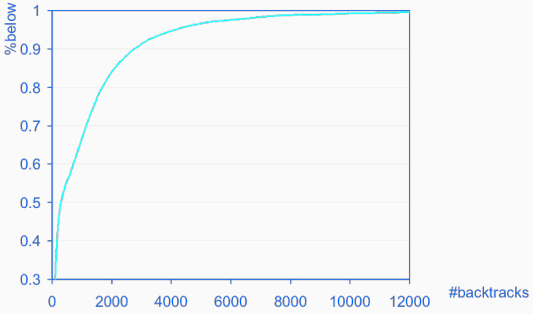
Learned clause: $(\bar{a} \vee \bar{r} \vee \bar{c} \vee \bar{d} \vee \bar{g})$

Minimized clause: $(\bar{a} \vee \bar{r} \vee \bar{c})$

Target	Curr Var	Marked	Unmarked	Vars to Trace	Action
r	r	$\{a, c\}$	$\emptyset$	$[a, b]$	–
r	a	$\{a, c\}$	$\emptyset$	$[b]$	a marked
r	b	$\{a, c\}$	$\{b\}$	$[]$	b decision & unmarked
r	–	$\{a, c\}$	$\{b\}$	$[]$	unmarked vars; $\therefore$ keep r
a, c	–	–	$\emptyset$	$[]$	a, c decision variables; keep both

Search Restarts

Branch randomization

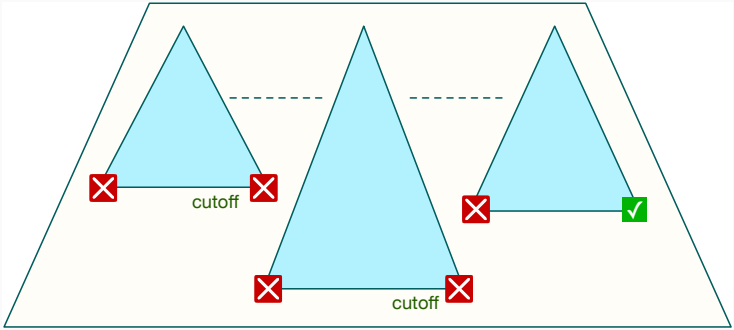


- Heavy-tail behavior: [GSC97]
  - 10000 runs, branching randomization on satisfiable industrial instance

$\therefore$ use rapid randomized restarts (search restarts)

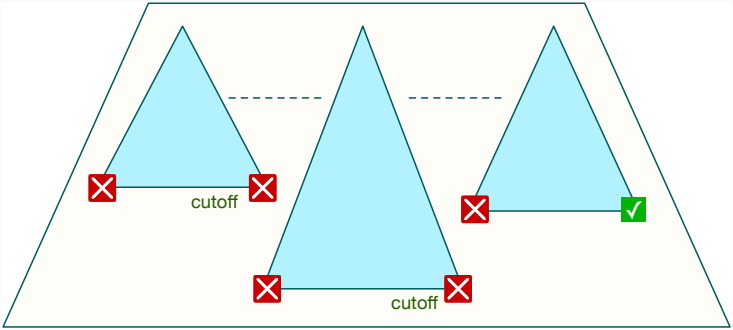
Search restarts

- Restart search after a number of conflicts



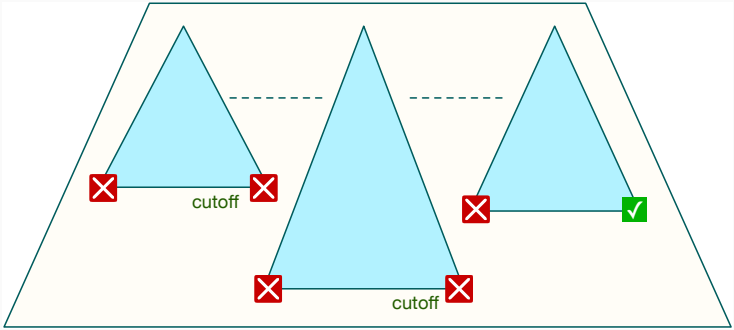
Search restarts

- Restart search after a number of conflicts
  - Increase cutoff after each restart
    - Guarantees completeness
    - Different policies exist



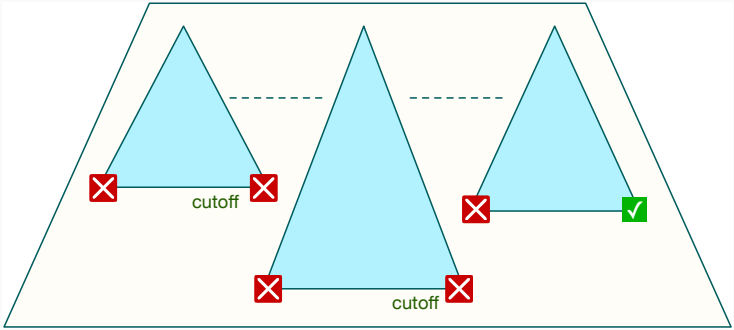
Search restarts

- Restart search after a number of conflicts
  - Increase cutoff after each restart
    - Guarantees completeness
    - Different policies exist
  - Effective for SAT & UNSAT formulas. **Why?**



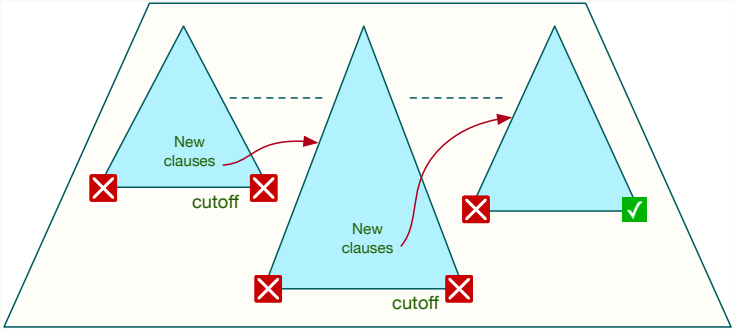
Search restarts

- Restart search after a number of conflicts
  - Increase cutoff after each restart
    - Guarantees completeness
    - Different policies exist
  - Effective for SAT & UNSAT formulas. **Why?**
    - Proof complexity arguments



Search restarts

- Restart search after a number of conflicts
  - Increase cutoff after each restart
    - Guarantees completeness
    - Different policies exist
  - Effective for SAT & UNSAT formulas. **Why?**
    - Proof complexity arguments
  - Clause learning (very) effective in between restarts



Lazy Data Structures

Data structures basics

- Recap states of a clause: unresolved, unit, falsified, satisfied
- Each literal l should access clauses containing l and $\bar{l}$
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  - Watched literals are one example of lazy data structures
    - But there are others [ZS00]

Watched literals

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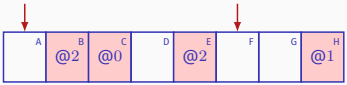

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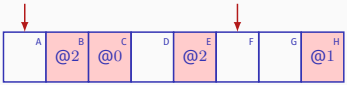

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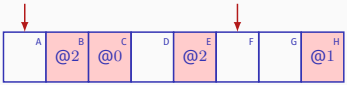

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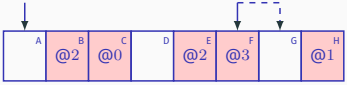

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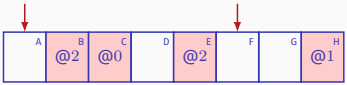

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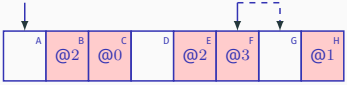

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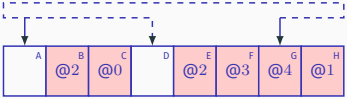

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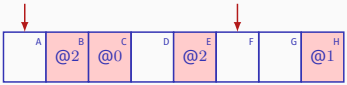

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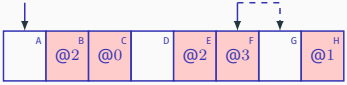

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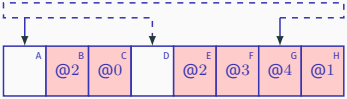

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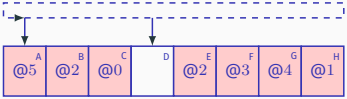

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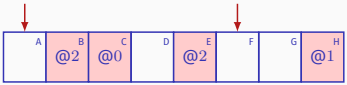

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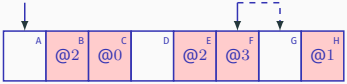

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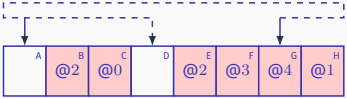

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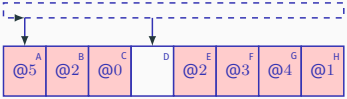

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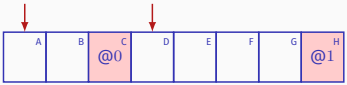

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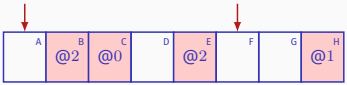
Watched literals – different implementations exist!



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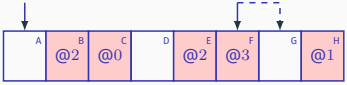

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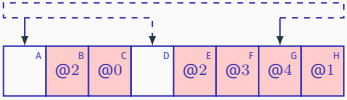

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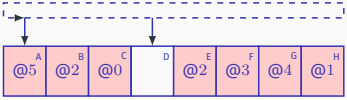

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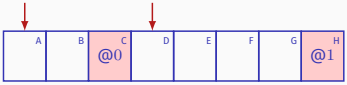

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  - Phase saving [PD07]
  - Novel restart strategies [Hua07, BF15, LOM+18]
  - Preprocessing/inprocessing [JHB12, HJL+15]
  - Clause minimization: LBD-based and UP-based [AS09, LLX+17]

Why CDCL Works?

Why CDCL works – a practitioner's view

- GRASP-like clause learning extensively inspired in circuit reasoners
  - UIPs mimic unique sensitization points (USPs), from testing
  - Analysis of conflicts organized by decision levels
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  - **Any** learned clause contains explanation given working assumptions (more next)

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$$\mathcal{B} = \{(\bar{a} \vee b), (\bar{a} \vee c)\}$$
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- E.g. given assumptions $\{s_1 = 1, s_2 = 0, s_3 = 0, s_4 = 1\}$, SAT solver handles formula:

$$\mathcal{F} = \{(\bar{a} \vee b), (\bar{a} \vee c), (a), (a \vee \bar{b})\}$$

which is satisfiable

Quiz – what happens in this case?

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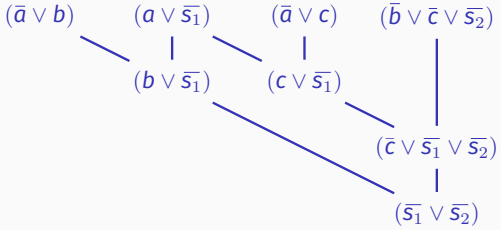

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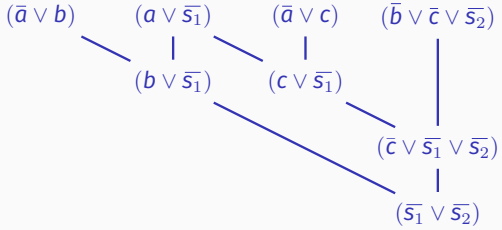

- Unsatisfiable core: 1^{st} and 2^{nd} clauses of $\mathcal{S}$, given $\mathcal{B}$

3 Modeling with SAT

Recap Clausification of Boolean Formulas

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- Standard textbook solution
  - Operator elimination; De Morgan's laws, remove double negations & apply distributivity
  - Worst-case exponential
  - Set of variables constant

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- **Obs:** *There are no CNF formulas* [Stu13]

- Standard textbook solution
  - Operator elimination; De Morgan's laws, remove double negations & apply distributivity
  - Worst-case exponential
  - Set of variables constant

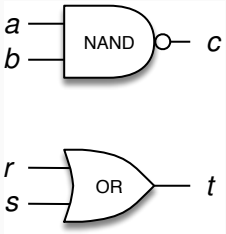
- Tseitin's translation & variants (next)
  - New variables added
  - Satisfiability is preserved
  - Linear size transformation

Representing Boolean formulas / circuits I

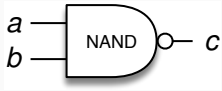
- Satisfiability problems can be defined on Boolean circuits/formulas
  - Can use any logic connective: $\wedge, \vee, \neg, \rightarrow, \leftrightarrow, \ldots$
- Can represent circuits/formulas as CNF formulas [Tse68, PG86]
  - For each (simple) gate, CNF formula encodes the consistent assignments to the gate's inputs and output
    - Given $z = \mathrm{OP}(x, y)$, represent in CNF $z \leftrightarrow \mathrm{OP}(x, y)$
  - CNF formula for the circuit is the conjunction of CNF formula for each gate

$$\mathcal{F}_c = (a \vee c) \wedge (b \vee c) \wedge (\bar{a} \vee \bar{b} \vee \bar{c})$$

$$\mathcal{F}_t = (\bar{r} \vee t) \wedge (\bar{s} \vee t) \wedge (r \vee s \vee \bar{t})$$



Representing Boolean formulas / circuits II



ab / c	00	01	11	10
0	0	0	1	0
1	1	1	0	1

a	b	c	$\mathcal{F}_c$(a,b,c)
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
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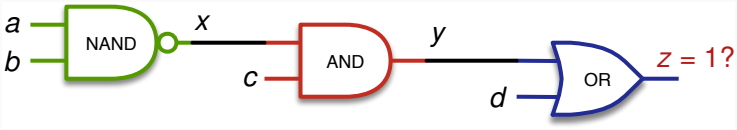
$$\mathcal{F}_c = (a \vee c) \wedge (b \vee c) \wedge (\bar{a} \vee \bar{b} \vee \bar{c})$$

Representing Boolean formulas / circuits III

- CNF formula for the circuit is the conjunction of the CNF formula for each gate
  - Can specify objectives with additional clauses



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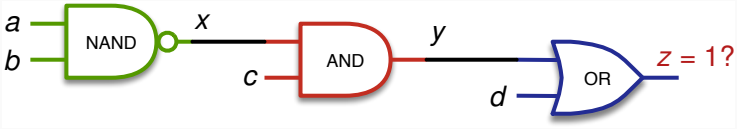

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- Note: $z = d \vee (c \wedge (\neg(a \wedge b)))$
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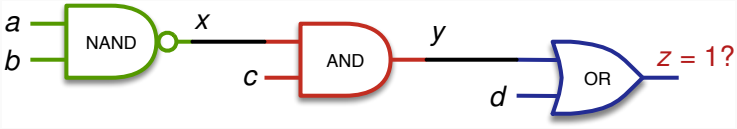

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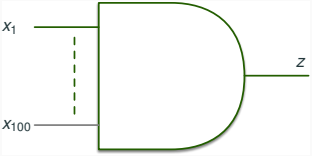
- Note: $z = d \vee (c \wedge (\neg(a \wedge b)))$
  - **No** distinction between Boolean circuits and (non-clausal) formulas, besides adding new variables
- Easy to do more structures: ITEs; Adders; etc.

Quiz – how to encode a 100 input gate?



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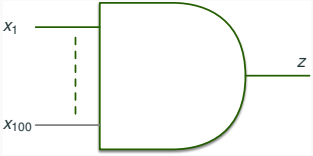

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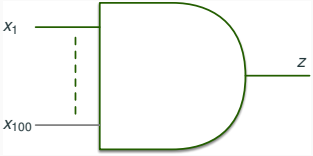

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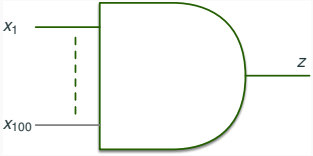

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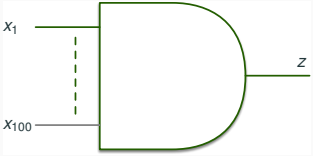

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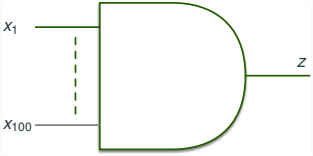

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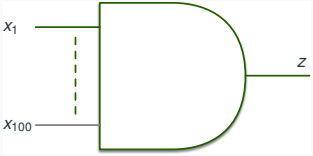

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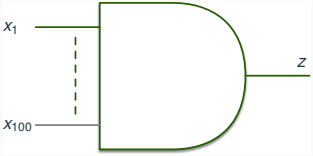

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- Similar ideas apply for other (simple) logical operators: AND, NAND, OR, NOR, etc.

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  - How to model linear cost function optimization?

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  - Hard constraints: φ
  - Soft constraints: $(\overline{x_j})$, each with cost c_j

Linear Constraints

Linear constraints

- Cardinality constraints: $\sum_{j=1}^{n} x_j \leq k$?
  - How to handle AtMost1 constraints, $\sum_{j=1}^{n} x_j \leq 1$?
  - General form: $\sum_{j=1}^{n} x_j \bowtie k$, with $\bowtie \in \{<, \leq, =, \geq, >\}$

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- If variables are non-Boolean, e.g. with finite domain
  - Need to encode variables (more later)

Equals1, AtLeast1 & AtMost1 constraints

- $\sum_{j=1}^{n} x_j = 1$: encode with $(\sum_{j=1}^{n} x_j \leq 1) \wedge (\sum_{j=1}^{n} x_j \geq 1)$
- $\sum_{j=1}^{n} x_j \geq 1$: encode with $(x_1 \vee x_2 \vee \ldots \vee x_n)$
- $\sum_{j=1}^{n} x_j \leq 1$ encode with:
  - Pairwise encoding
    - Clauses: $\mathcal{O}(n^2)$; No auxiliary variables
  - Sequential counter [Sin05]
    - Clauses: $\mathcal{O}(n)$; Auxiliary variables: $\mathcal{O}(n)$
  - Bitwise encoding [FP01, Pre07]
    - Clauses: $\mathcal{O}(n \log n)$; Auxiliary variables: $\mathcal{O}(\log n)$
  - ...

Pairwise encoding

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$$
\begin{aligned}
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b \to \bar{c} \wedge \bar{d} \wedge \bar{a} &\implies (\bar{b} \vee \bar{c}) \wedge (\bar{b} \vee \bar{d}) \wedge (\bar{b} \vee \bar{a}) \\
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\end{aligned}
$$

  - Encoded as: $(\bar{a} \vee \bar{b}) \wedge (\bar{a} \vee \bar{c}) \wedge (\bar{a} \vee \bar{d}) \wedge (\bar{b} \vee \bar{c}) \wedge (\bar{b} \vee \bar{d}) \wedge (\bar{c} \vee \bar{d})$

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- With N variables, number of clauses becomes $\frac{n(n-1)}{2}$
  - But no additional variables

Sequential counter encoding

- Encode $\sum_{j=1}^{n} x_j \leq 1$ with sequential counter:

$$(\bar{x}_1 \vee s_1) \wedge (\bar{x}_n \vee \bar{s}_{n-1}) \wedge$$
$$\bigwedge_{1<i<n} ((\bar{x}_i \vee s_i) \wedge (\bar{s}_{i-1} \vee s_i) \wedge (\bar{x}_i \vee \bar{s}_{i-1}))$$

  - If some $x_j = 1$, then all s_i variables must be assigned
    - $s_i = 1$ for $i \geq j$, and so $x_i = 0$ for $i > j$
    - $s_i = 0$ for $i < j$, and so $x_i = 0$ for $i < j$
    - Thus, all other x_i variables must take value 0
  - If all $x_j = 0$, can find consistent assignment to s_i variables
  - $\mathcal{O}(n)$ clauses ; $\mathcal{O}(n)$ auxiliary variables

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- Encode $\sum_{j=1}^{n} x_j \leq 1$ with bitwise encoding:

- An example: $x_1 + x_2 + x_3 \leq 1$

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  - Auxiliary variables $v_0, \ldots, v_{r-1}$; $r = \lceil \log n \rceil$ (with $n > 1$)
  - If $x_j = 1$, then $v_0 \ldots v_{r-1} = b_0 \ldots b_{r-1}$, the binary encoding of $j - 1$

    $x_j \rightarrow (v_0 = b_0) \wedge \ldots \wedge (v_{r-1} = b_{r-1}) \Leftrightarrow (\bar{x}_j \vee (v_0 = b_0) \wedge \ldots \wedge (v_{r-1} = b_{r-1}))$

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	$j-1$	$v_1 v_0$
x_1	0	00
x_2	1	01
x_3	2	10

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  - Clauses $(\bar{x}_j \vee (v_i \leftrightarrow b_i)) = (\bar{x}_j \vee l_i)$, $i = 0, \ldots, r-1$, where
    - $l_i \equiv v_i$, if $b_i = 1$
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    - $l_i \equiv v_i$, if $b_i = 1$
    - $l_i \equiv \bar{v}_i$, otherwise
  - If $x_j = 1$, assignment to v_i variables must encode $j - 1$
    - For consistency, all other x variables must not take value 1
  - If all $x_j = 0$, any assignment to v_i variables is consistent
  - $\mathcal{O}(n \log n)$ clauses ; $\mathcal{O}(\log n)$ auxiliary variables
- An example: $x_1 + x_2 + x_3 \leq 1$

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x_1	0	00
x_2	1	01
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$$(\bar{x}_2 \vee \bar{v}_1) \wedge (\bar{x}_2 \vee v_0)$$
$$(\bar{x}_3 \vee v_1) \wedge (\bar{x}_3 \vee \bar{v}_0)$$

General cardinality constraints

- General form: $\sum_{j=1}^{n} x_j \leq k$ (or $\sum_{j=1}^{n} x_j \geq k$)
  - Operational encoding [War98]
    - Clauses/Variables: $\mathcal{O}(n)$
    - Does not ensure arc-consistency
  - Generalized pairwise
    - Clauses: $\mathcal{O}(2^n)$; no auxiliary variables
  - Sequential counters [Sin05]
    - Clauses/Variables: $\mathcal{O}(n\,k)$
  - BDDs [ES06]
    - Clauses/Variables: $\mathcal{O}(n\,k)$
  - Sorting networks [Bat68, ES06]
    - Clauses/Variables: $\mathcal{O}(n \log^2 n)$
  - Cardinality Networks: [ANOR09, ANOR11]
    - Clauses/Variables: $\mathcal{O}(n \log^2 k)$
  - Pairwise Cardinality Networks: [CZ10]
  - ...

Generalized pairwise encoding

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  - Encoded as: $(\bar{a} \vee \bar{b} \vee \bar{c}) \wedge (\bar{a} \vee \bar{b} \vee \bar{d}) \wedge (\bar{a} \vee \bar{c} \vee \bar{d}) \wedge (\bar{b} \vee \bar{c} \vee \bar{d})$

Generalized pairwise encoding

- General form: $\sum_{j=1}^{n} x_j \leq k$
- Any combination of $k+1$ true variables is disallowed

- Example: $a + b + c + d \leq 2$

$$
\begin{aligned}
a \wedge b \rightarrow \bar{c} &\implies (\bar{a} \vee \bar{b} \vee \bar{c}) \\
a \wedge b \rightarrow \bar{d} &\implies (\bar{a} \vee \bar{b} \vee \bar{d}) \\
a \wedge c \rightarrow \bar{d} &\implies (\bar{a} \vee \bar{c} \vee \bar{d}) \\
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- In general, number of clauses is C_{k+1}^{n}
  - Recall: for AtMost1 (i.e. for $k = 1$), number of clauses is: $\frac{n(n-1)}{2}$

Another example

- Example: $a + b + c + d + e \leq 2$
- Encoding will contain $C_3^5 = 10$ clauses:

$$
\begin{aligned}
a \wedge b \rightarrow \bar{c} &\implies (\bar{a} \vee \bar{b} \vee \bar{c}) \\
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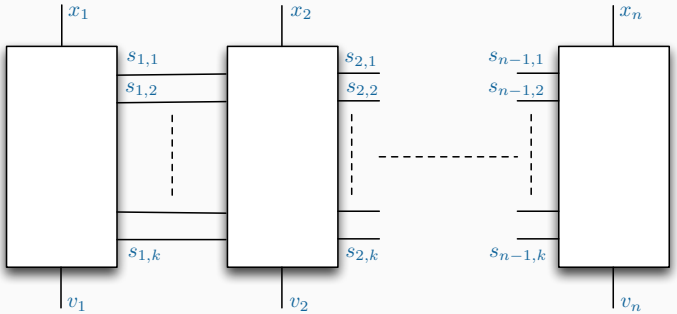
\end{aligned}
$$

Sequential counter – revisited I

- Encode $\sum_{j=1}^{n} x_j \leq k$ with sequential counter:



- Equations for each block $1 < i < n$, $1 < j < k$:

$s_i = \sum_{j=1}^{i} x_j$
s_i represented in unary

$s_{i,1} = s_{i-1,1} \vee x_i$
$s_{i,j} = s_{i-1,j} \vee s_{i-1,j-1} \wedge x_i$
$v_i = (s_{i-1,k} \wedge x_i) = 0$

Sequential counter – revisited II

- CNF formula for $\sum_{j=1}^{n} x_j \leq k$:
  - Assume: $k > 0 \land n > 1$
  - Indeces: $1 < i < n\ ,\ 1 < j \leq k$

$$
\begin{aligned}
&(\neg x_1 \lor s_{1,1}) \\
&(\neg s_{1,j}) \\
&(\neg x_i \lor s_{i,1}) \\
&(\neg s_{i-1,1} \lor s_{i,1}) \\
&(\neg x_i \lor \neg s_{i-1,j-1} \lor s_{i,j}) \\
&(\neg s_{i-1,j} \lor s_{i,j}) \\
&(\neg x_i \lor \neg s_{i-1,k}) \\
&(\neg x_n \lor \neg s_{n-1,k})
\end{aligned}
$$

- $\mathcal{O}(n\ k)$ clauses & variables

Pseudo-Boolean constraints

- General form: $\sum_{j=1}^{n} a_j x_j \leq b$
  - Operational encoding [War98]
    - Clauses/Variables: $\mathcal{O}(n)$
    - Does not guarantee arc-consistency
  - BDDs [ES06]
    - Worst-case exponential number of clauses

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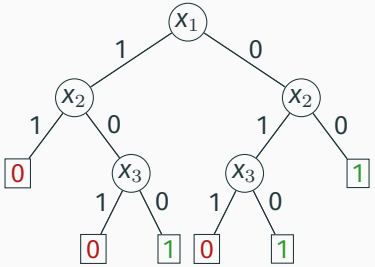
  - Improved polynomial watchdog encoding [ANO+12]
    - Clauses & aux variables: $\mathcal{O}(n^3 \log(a_{max}))$
  - ...

Encoding PB constraints with BDDs I

- Encode $3x_1 + 3x_2 + x_3 \leq 3$
- Construct BDD
  - E.g. analyze variables by decreasing coefficients
- Extract ITE-based circuit from BDD



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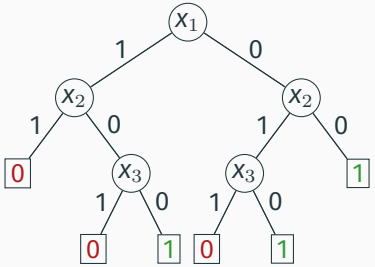
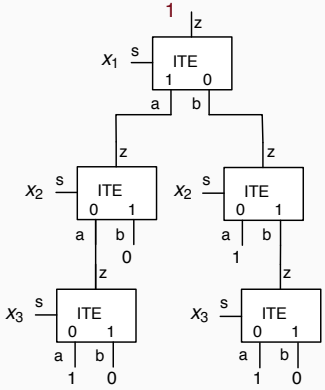
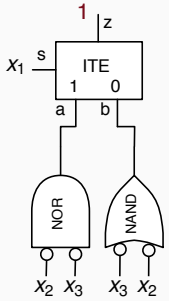




Encoding PB constraints with BDDs II

- Encode $3x_1 + 3x_2 + x_3 \leq 3$
- Extract ITE-based circuit from BDD
- Simplify and create final circuit:



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  - Let $x_2 = 0$
  - Either constraint can still be satisfied, but **not** both

Encoding CSPs

CSP constraints

- Many possible encodings:
  - Direct encoding [dK89, GJ96, Wal00]
  - Log encoding [Wal00]
  - Support encoding [Kas90, Gen02]
  - Log-Support encoding [Gav07]
  - Order encoding for finite linear CSPs [TTKB09]

Direct encoding for CSP w/ binary constraints

- Variable x_i with domain D_i, with $m_i = |D_i|$
- Constraints are relations over domains of variables
  - For a constraint over $x_1, \ldots, x_k$, define relation $R \subseteq D_1 \times \cdots \times D_k$
  - Need to encode elements not in the relation
  - For a binary relation, use set of binary clauses, one for each element not in R
- Represent values of x_i with Boolean variables $x_{i,1}, \ldots, x_{i,m_i}$
- Require $\sum_{k=1}^{m_i} x_{i,k} = 1$
  - Suffices to require $\sum_{k=1}^{m_i} x_{i,k} \geq 1$ [Wal00]
- If the pair of assignments $x_i = v_i \wedge x_j = v_j$ is not allowed, add binary clause $(\bar{x}_{i,v_i} \vee \bar{x}_{j,v_j})$

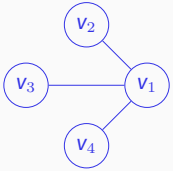
Additional topics

- Encoding problems to SAT is ubiquitous:
  - Many more encodings of finite domain CSP into SAT
  - Encodings of Answer Set Programming (ASP) into SAT
  - Eager SMT solving
  - Theorem provers iteratively encode problems into SAT
  - Model finders interatively encode problems into SAT
  - ...

Modeling Examples

Minimum vertex cover

- The problem:
  - Graph $G = (V, E)$
  - Vertex cover $U \subseteq V$
    - For each $(v_i, v_j) \in E$, either $v_i \in U$ or $v_j \in U$
  - Minimum vertex cover: vertex cover U of minimum size



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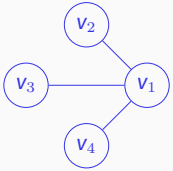

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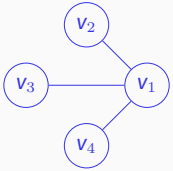

Vertex cover: $\{v_2, v_3, v_4\}$

Min vertex cover: $\{v_1\}$

Minimum vertex cover

- Modeling with Pseudo-Boolean Optimization (PBO):
  - Variables: x_i for each $v_i \in V$, with $x_i = 1$ iff $v_i \in U$
  - Clauses: $(x_i \vee x_j)$ for each $(v_i, v_j) \in E$
  - Objective function: minimize number of true x_i variables
    - I.e. minimize vertices included in U

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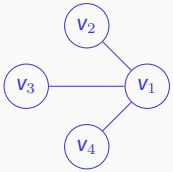

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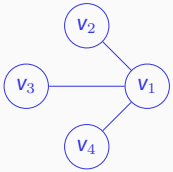

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- Alternative propositional encoding:

$$\begin{aligned} \varphi_S &= \{(\neg x_1), (\neg x_2), (\neg x_3), (\neg x_4)\} \\ \varphi_H &= \{(x_1 \vee x_2), (x_1 \vee x_3), (x_1 \vee x_4)\} \end{aligned}$$

Graph coloring

- Given undirected graph $G = (V, E)$ and k colors:
  - Can we assign colors to vertices of G s.t. any pair of adjacent vertices are assigned different colors?

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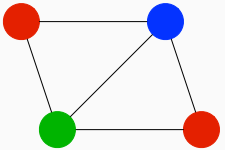
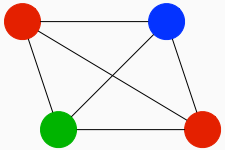


Valid coloring



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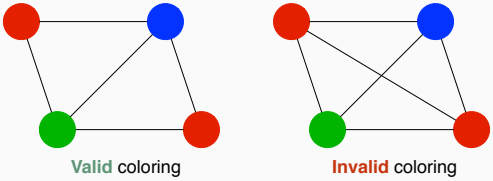

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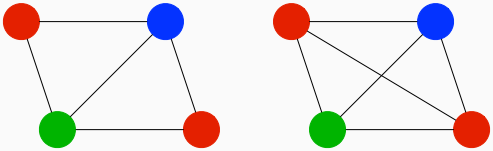

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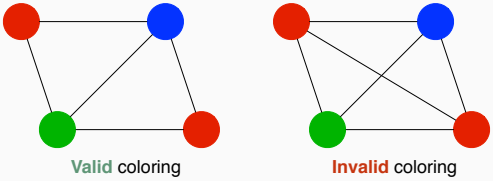

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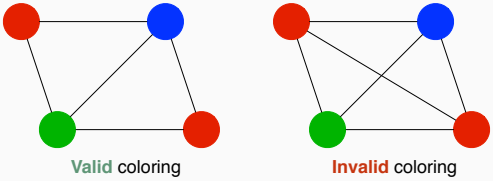

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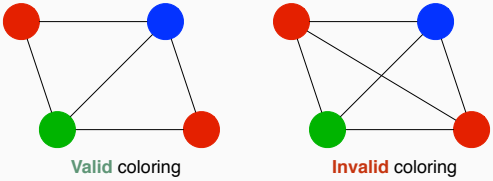

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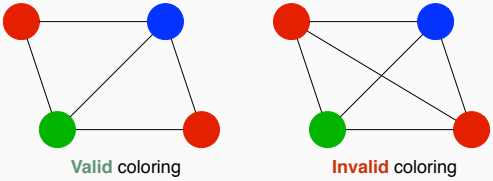

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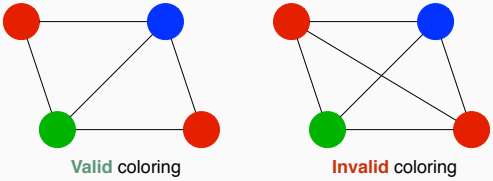
  - $\sum_{j \in \{1,\ldots,k\}} x_{i,j} = 1$**, for** $v_i \in V$

Graph coloring

- Given undirected graph $G = (V, E)$ and k colors:
  - Can we assign colors to vertices of G s.t. any pair of adjacent vertices are assigned different colors?



- How to model color assignments to vertices?
  - $x_{i,j} = 1$ **iff vertex** $v_i \in V$ **is assigned color** $j \in \{1, \ldots, k\}$
- How to model adjacent vertices with different colors?
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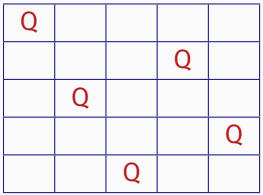
- How to model vertices get some color?
  - $\sum_{j \in \{1,\ldots,k\}} x_{i,j} = 1$**, for** $v_i \in V$
  - **Note: it suffices to use** $\left(\bigvee_{j \in \{1,\ldots,k\}} x_{i,j}\right)$

The N-Queens problem I

- The N-Queens Problem:
  Place N queens on a $N \times N$ board, such that no two queens attack each other

- Example for a 5×5 board:



The N-Queens problem II

- x_{ij}: 1 if queen placed in position (i, j); 0 otherwise
- Each row must have exactly one queen:

$$1 \leq i \leq N, \qquad \sum_{j=1}^{N} x_{ij} = 1$$

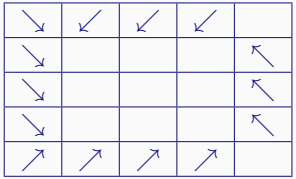
- Each column must have exactly one queen:

$$1 \leq j \leq N, \qquad \sum_{i=1}^{N} x_{ij} = 1$$

- Also, need to define constraints on diagonals...

The N-Queens problem III

- Each diagonal can have at most one queen:



$$i = 1, \qquad 2 \leq j < N, \qquad \sum_{k=0}^{j-1} x_{i+k\, j-k} \leq 1$$

$$i = N, \qquad 1 \leq j < N, \qquad \sum_{k=0}^{N-j} x_{i-k\, j+k} \leq 1$$

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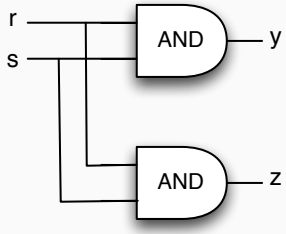
Design debugging

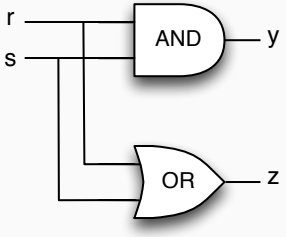
[SMV+07]

Correct circuit



Input stimuli: $\langle r, s \rangle = \langle 0, 1 \rangle$
Valid output: $\langle y, z \rangle = \langle 0, 0 \rangle$

Faulty circuit



Input stimuli: $\langle r, s \rangle = \langle 0, 1 \rangle$
Invalid output: $\langle y, z \rangle = \langle 0, 0 \rangle$

- The model:
  - Hard clauses: Input and output values
  - Soft clauses: CNF representation of circuit
- The problem:
  - Maximize number of satisfied clauses (i.e. circuit gates)

Software package upgrades

[MBC+06, TSJL07, AL08, ALS09, ABL+10b]

- Universe of software packages: $\{p_1, \ldots, p_n\}$
- Associate x_i with p_i: $x_i = 1$ iff p_i is installed
- Constraints associated with package p_i: (p_i, D_i, C_i)
  - D_i: dependencies (required packages) for installing p_i
  - C_i: conflicts (disallowed packages) for installing p_i
- Example problem: Maximum Installability
  - Maximum number of packages that can be installed
  - Package constraints represent hard clauses
  - Soft clauses: (x_i)

Package constraints:

$(p_1, \{p_2 \vee p_3\}, \{p_4\})$
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MaxSAT formulation:

$$\begin{aligned} \varphi_H &= \{(\neg x_1 \vee x_2 \vee x_3), (\neg x_1 \vee \neg x_4), \\ &\quad (\neg x_2 \vee x_3), (\neg x_2 \vee \neg x_4), (\neg x_3 \vee x_2), \\ &\quad (\neg x_4 \vee x_2), (\neg x_4 \vee x_3)\} \\ \varphi_S &= \{(x_1), (x_2), (x_3), (x_4)\} \end{aligned}$$

The knapsack problem

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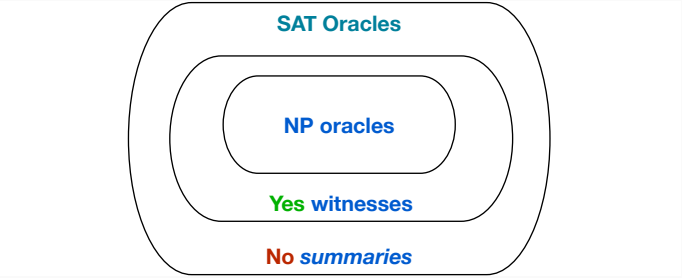
- Propositional encoding for the knapsack problem?

- **Solution:** consider 0-1 ILP (or PBO) formulation:
  - Associate propositional variable x_i with each objet i
  - $x_i = 1$ iff object i is picked

$$\begin{array}{ll} \max & \sum_{i=1}^{n} v_i \cdot x_i \\ \text{s.t} & \sum_{i=1}^{n} w_i \cdot x_i \leq W \end{array}$$

4 Problem Solving with SAT Oracles

So what are SAT oracles?



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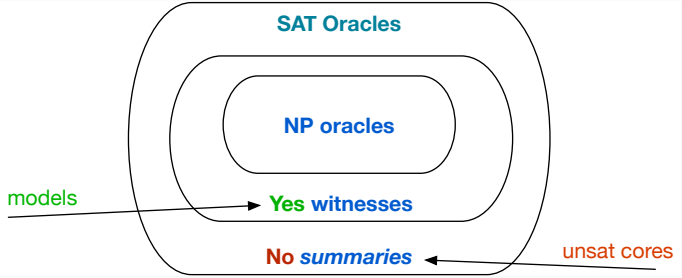

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- FSAT is an example of a function problem
  - **Note:** FSAT can be solved with **one** SAT oracle call

Beyond decision problems

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Yes/No	Decision Problems

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Some solution	

Beyond decision problems

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Beyond decision problems

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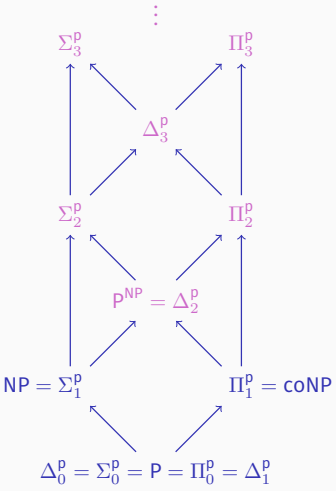
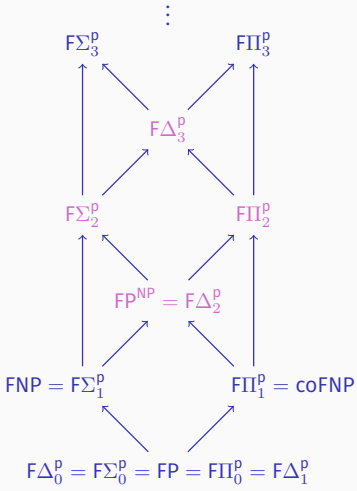
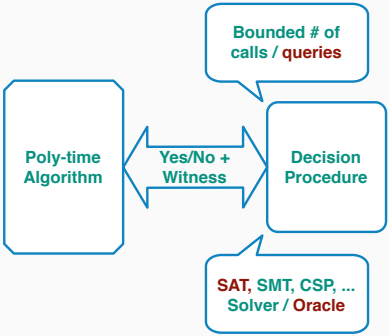
# solutions	Counting Problems

… and beyond NP – decision and function problems

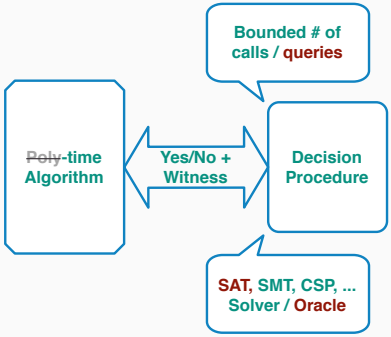




Oracle-based problem solving – simple scenario



Oracle-based problem solving – general setting



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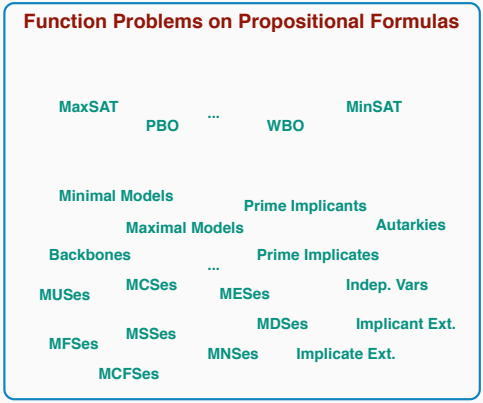

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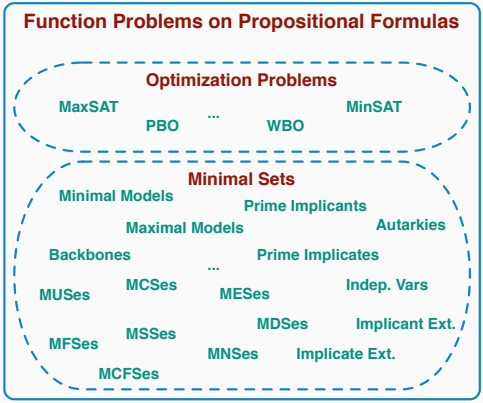
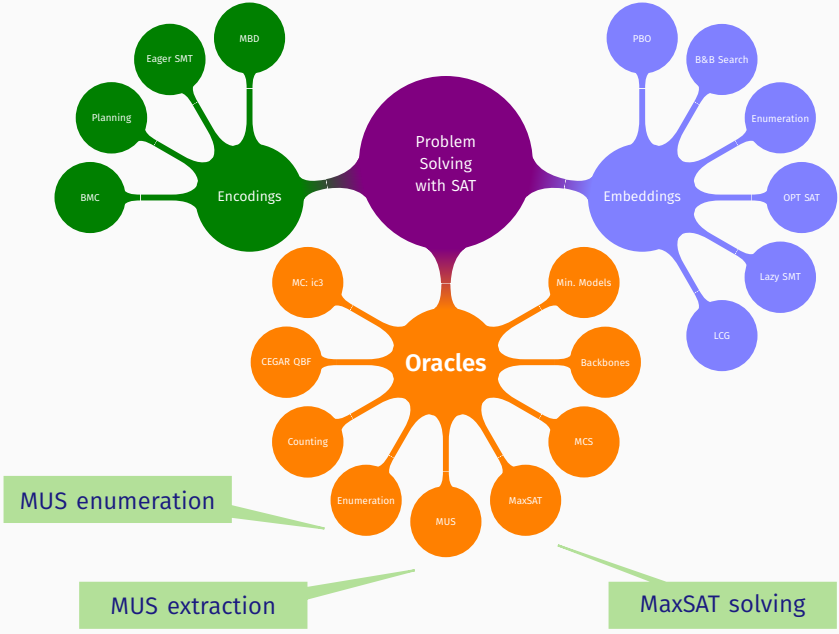


Selection of topics



Minimal Unsatisfiability

Analyzing inconsistency – timetabling

Subject	Day	Time	Room
Intro Prog	Mon	9:00-10:00	6.2.46
Intro AI	Tue	10:00-11:00	8.2.37
Databases	Tue	11:00-12:00	8.2.37
... (hundreds of consistent constraints)			
Linear Alg	Mon	9:00-10:00	6.2.46
Calculus	Tue	10:00-11:00	8.2.37
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- Given $\mathcal{F}$ $(\models \bot)$, $\mathcal{C} \subseteq \mathcal{F}$ is a Minimal Correction Subset (MCS) iff $\mathcal{F} \setminus \mathcal{C} \not\models \bot$ and $\forall_{\mathcal{C}' \subsetneq \mathcal{C}}, \mathcal{F} \setminus \mathcal{C}' \models \bot$. $\mathcal{S} = \mathcal{F} \setminus \mathcal{C}$ is MSS

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  - Easy to see **why**
- How to compute MUSes & MCSes **efficiently** with SAT oracles?

Why it matters?

- Analysis of over-constrained systems
  - Model-based diagnosis [Rei87]
    - Software fault localization
    - Spreadsheet debugging
    - Debugging relational specifications (e.g. Alloy)
    - Type error debugging
    - Axiom pinpointing in description logics
    - ...
  - Model checking of software & hardware systems
  - Inconsistency measurement
  - Minimal models; MinCost SAT; ...
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- Find minimal relaxations to recover consistency
  - But also minimum relaxations to recover consistency, eg. **MaxSAT**
- **Find minimal explanations of inconsistency**
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Deletion-based algorithm

Input : Set $\mathcal{F}$
Output: Minimal subset $\mathcal{M}$
begin
  $\mathcal{M} \leftarrow \mathcal{F}$
  foreach $c \in \mathcal{M}$ **do**
    if $\neg\mathsf{SAT}(\mathcal{M} \setminus \{c\})$ **then**
      $\mathcal{M} \leftarrow \mathcal{M} \setminus \{c\}$ // If $\neg\mathsf{SAT}(\mathcal{M} \setminus \{c\})$, then $c \notin$ MUS
  return $\mathcal{M}$ // Final $\mathcal{M}$ is MUS
end

- Number of oracles calls: $\mathcal{O}(m)$

[CD91, BDTW93]

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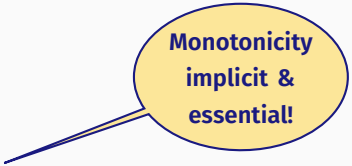
  foreach $c \in \mathcal{M}$ **do**
    if $\neg\mathsf{SAT}(\mathcal{M} \setminus \{c\})$ **then**
      $\mathcal{M} \leftarrow \mathcal{M} \setminus \{c\}$ // Remove c from $\mathcal{M}$
  return $\mathcal{M}$ // Final $\mathcal{M}$ is MUS
end



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[CD91, BDTW93]

Deletion – MUS example

c_1	c_2	c_3	c_4	c_5	c_6	c_7
$(\neg x_1 \vee \neg x_2)$	(x_1)	(x_2)	$(\neg x_3 \vee \neg x_4)$	(x_3)	(x_4)	$(x_5 \vee x_6)$

$\mathcal{M}$	$\mathcal{M} \setminus \{c\}$	$\neg\mathsf{SAT}(\mathcal{M} \setminus \{c\})$	Outcome

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$c_1..c_7$	$c_2..c_7$	1	Drop c_1

Deletion – MUS example

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$(\neg x_1 \vee \neg x_2)$	(x_1)	(x_2)	$(\neg x_3 \vee \neg x_4)$	(x_3)	(x_4)	$(x_5 \vee x_6)$

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$c_4..c_7$	$c_5..c_7$	0	Keep c_4

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c_1	c_2	c_3	c_4	c_5	c_6	c_7
$(\neg x_1 \vee \neg x_2)$	(x_1)	(x_2)	$(\neg x_3 \vee \neg x_4)$	(x_3)	(x_4)	$(x_5 \vee x_6)$

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$c_4..c_7$	$c_5..c_7$	0	Keep c_4
$c_4..c_7$	$c_4c_6c_7$	0	Keep c_5

Deletion – MUS example

c_1	c_2	c_3	c_4	c_5	c_6	c_7
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$c_4..c_7$	$c_4..c_6$	1	Drop c_7

Deletion – MUS example

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$c_4..c_7$	$c_4..c_6$	1	Drop c_7

- MUS: $\{c_4, c_5, c_6\}$

Many MUS algorithms

- Formula $\mathcal{F}$ with m clauses k the size of largest minimal subset

Algorithm	Oracle Calls	Reference
Insertion-based	$\mathcal{O}(k\,m)$	[dSNP88, vMW08]
MCS_MUS	$\mathcal{O}(k\,m)$	[BK15]
Deletion-based	$\mathcal{O}(m)$	[CD91, BDTW93]
Linear insertion	$\mathcal{O}(m)$	[MSL11, BLM12]
Dichotomic	$\mathcal{O}(k\,\log(m))$	[HLSB06]
QuickXplain	$\mathcal{O}(k + k\,\log(\frac{m}{k}))$	[Jun04]
Progression	$\mathcal{O}(k\,\log(1+\frac{m}{k}))$	[MJB13]

- **Note:** Lower bound in $\mathrm{FP}^{\mathrm{NP}}_{||}$ and upper bound in $\mathrm{FP}^{\mathrm{NP}}$ [CT95]
- Oracle calls correspond to testing unsatisfiability with SAT solver
- Practical optimizations: clause set trimming; clause set refinement; redundancy removal; (recursive) model rotation

MUS Enumeration

How to enumerate MUSes?

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1. Standard solution:
   Exploit HS duality between MCSes and MUSes [Rei87, LS08]

   MCSes are MHSes of MUSes and vice-versa

   - Enumerate **all** MCSes and then enumerate **all** MHSes of the MCSes, i.e. compute all the MUSes
   - Problematic if too many MCSes, and we want the MUSes
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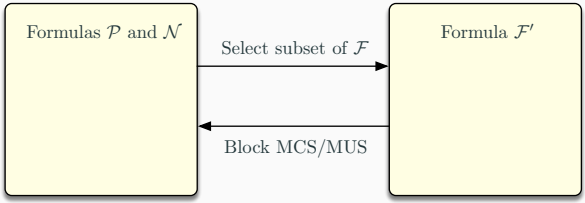
2. Exploit recent advances in 2QBF solving

3. Implicit hitting set dualization [LPMM16]
   - Most effective if MUSes provided to user on-demand

How to enumerate MUSes, preferably?



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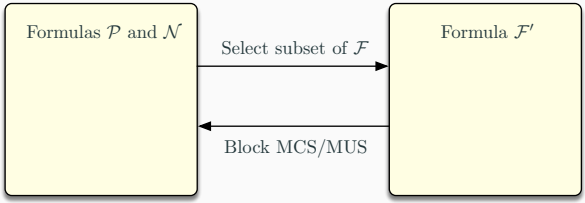

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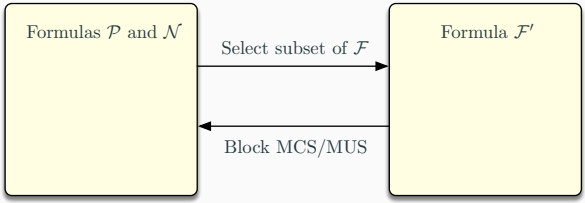

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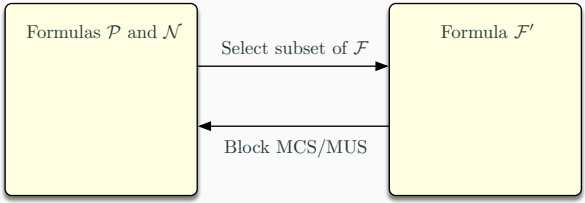

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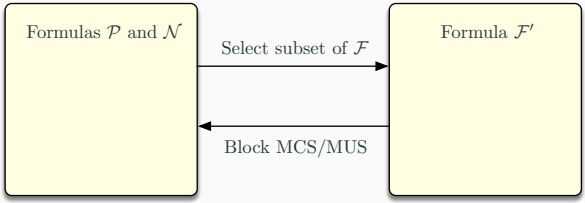

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5. Repeat loop

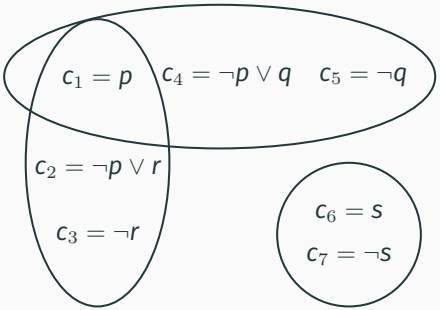
MARCO/eMUS algorithm

Input: CNF formula $\mathcal{F}$

```
1  begin
2      I ← {p_i | c_i ∈ F}
3      (P, N) ← (∅, ∅)
4      while true do
5          (st, H) ← MinHittingSet(N, P)
6          if not st then return
7          F' ← {c_i | p_i ∈ I ∧ p_i ∉ H}
8          if not SAT(F') then
9              M ← ComputeMUS(F')
10             ReportMUS (M)
11             N ← N ∪ {¬p_i | c_i ∈ M}
12         else
13             P ← P ∪ {p_i | p_i ∈ H}
14 end
```

An example

MinHS ($\mathcal{N}$) $p_1p_2p_3p_4p_5p_6p_7$	$\mathcal{F}'$ S/U	MUS/MCS
1111111	U	$\neg p_1 \vee \neg p_2 \vee \neg p_3$
0111111	U	$\neg p_6 \vee \neg p_7$
0111101	S	$p_1 \vee p_6$
1011101	U	$\neg p_1 \vee \neg p_4 \vee \neg p_5$
1101010	S	$p_3 \vee p_5 \vee p_7$
1010110	S	$p_2 \vee p_4 \vee p_7$
1100101	S	$p_3 \vee p_4 \vee p_6$
0111110	S	$p_1 \vee p_7$
1101001	S	$p_3 \vee p_5 \vee p_6$
1010101	S	$p_2 \vee p_4 \vee p_6$
1011001	S	$p_2 \vee p_5 \vee p_6$
1100110	S	$p_3 \vee p_4 \vee p_7$
1011010	S	$p_2 \vee p_5 \vee p_7$



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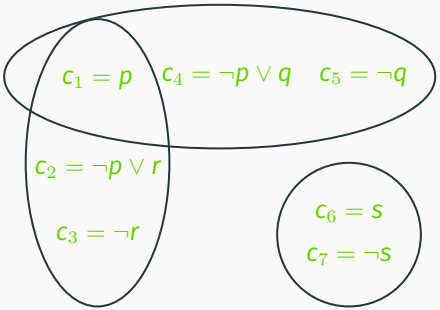

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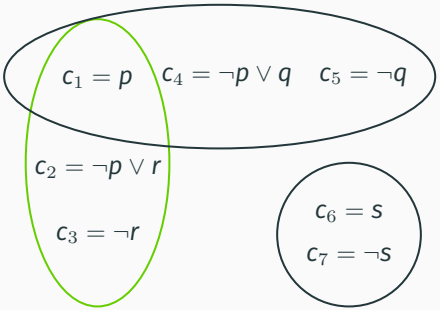
1100110	S	$p_3 \vee p_4 \vee p_7$
1011010	S	$p_2 \vee p_5 \vee p_7$



An example

MinHS ($\mathcal{N}$) $p_1p_2p_3p_4p_5p_6p_7$	$\mathcal{F}'$ S/U	MUS/MCS
1111111	U	$\neg p_1 \vee \neg p_2 \vee \neg p_3$
0111111	U	$\neg p_6 \vee \neg p_7$
0111101	S	$p_1 \vee p_6$
1011101	U	$\neg p_1 \vee \neg p_4 \vee \neg p_5$
1101010	S	$p_3 \vee p_5 \vee p_7$
1010110	S	$p_2 \vee p_4 \vee p_7$
1100101	S	$p_3 \vee p_4 \vee p_6$
0111110	S	$p_1 \vee p_7$
1101001	S	$p_3 \vee p_5 \vee p_6$
1010101	S	$p_2 \vee p_4 \vee p_6$
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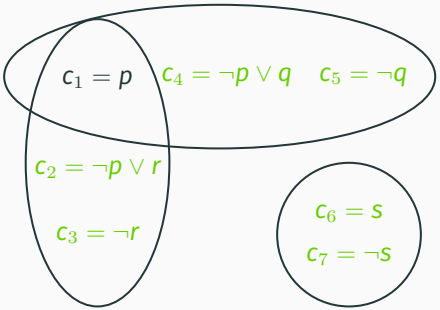

An example

MinHS ($\mathcal{N}$) $p_1p_2p_3p_4p_5p_6p_7$	$\mathcal{F}'$ S/U	MUS/MCS
1111111	U	$\neg p_1 \vee \neg p_2 \vee \neg p_3$
0111111	U	$\neg p_6 \vee \neg p_7$
0111101	S	$p_1 \vee p_6$
1011101	U	$\neg p_1 \vee \neg p_4 \vee \neg p_5$
1101010	S	$p_3 \vee p_5 \vee p_7$
1010110	S	$p_2 \vee p_4 \vee p_7$
1100101	S	$p_3 \vee p_4 \vee p_6$
0111110	S	$p_1 \vee p_7$
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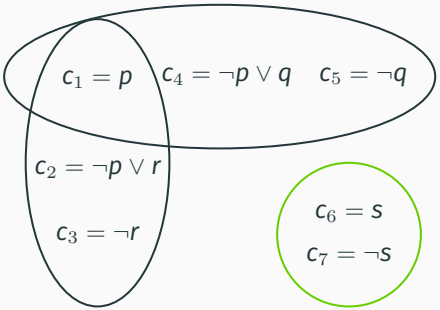

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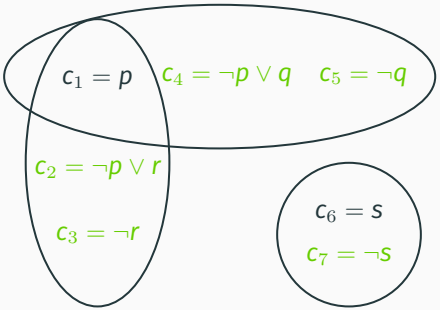

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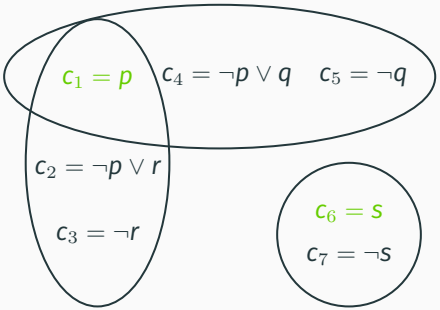
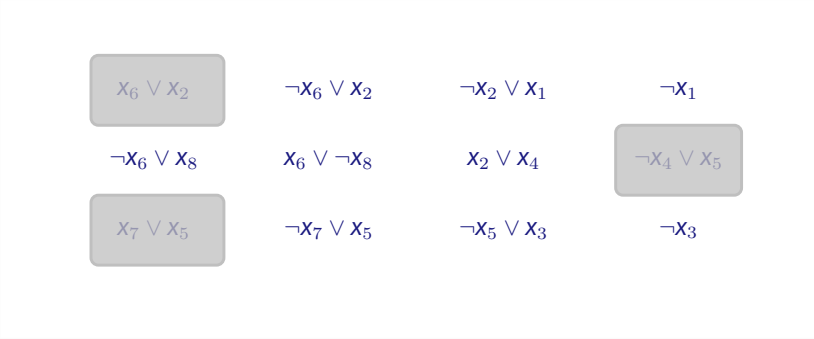


Maximum Satisfiability

Recap MaxSAT

- Given unsatisfiable formula, find largest subset of clauses that is satisfiable

Recap MaxSAT



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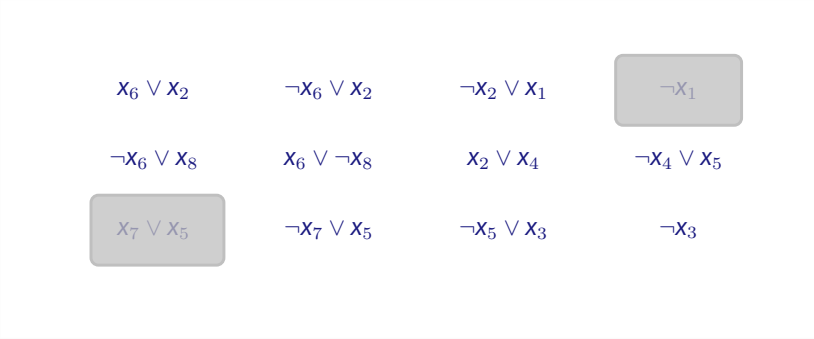

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- The MaxSAT solution is one of the smallest MCSes

Recap MaxSAT

$$x_6 \vee x_2 \qquad \neg x_6 \vee x_2 \qquad \neg x_2 \vee x_1 \qquad \neg x_1$$

$$\neg x_6 \vee x_8 \qquad x_6 \vee \neg x_8 \qquad x_2 \vee x_4 \qquad \neg x_4 \vee x_5$$

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  - **Note**: Clauses can have weights & there can be hard clauses
- **Many** practical applications

[SZGN17]

MaxSAT problem(s)

		Hard Clauses?	
		No	Yes
Weights?	No		
	Yes		

MaxSAT problem(s)

		Hard Clauses?	
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Weights?	No	Plain	Partial
	Yes	Weighted	Weighted Partial

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- Compute set of satisfied soft clauses with maximum cost
  - Without weights, cost of each falsified soft clause is 1
- **Or**, compute set of falsified soft clauses with minimum cost
  (s.t. hard & remaining soft clauses are satisfied)

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- **Or**, compute set of falsified soft clauses with minimum cost
  (s.t. hard & remaining soft clauses are satisfied)
- **Note**: goal is to compute **set** of satisfied (or falsified) clauses;
  not just the cost **!**

Issues with MaxSAT

- **Unit propagation is unsound for MaxSAT**

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  - Formula with all clauses soft:

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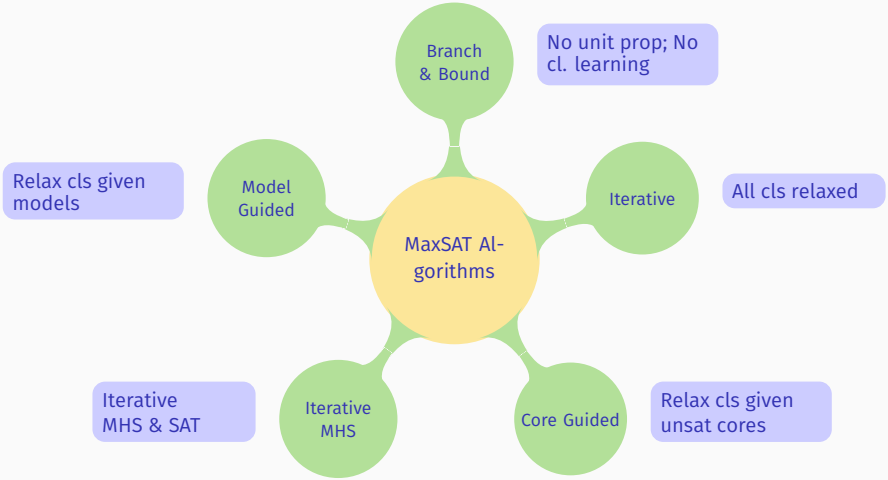
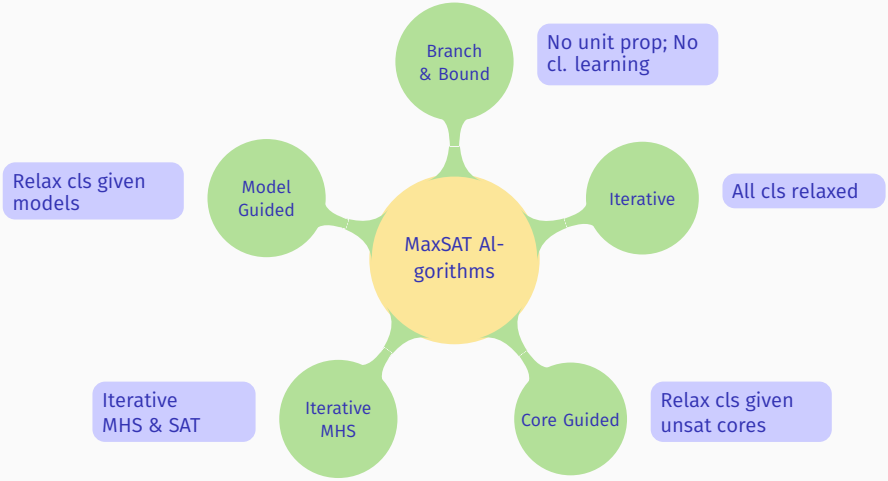
  - Is 2 the MaxSAT solution??
  - **No!** Enough to either falsify (x) or (z)
- Cannot use unit propagation
- Cannot learn clauses (using unit propagation)
- Need to solve MaxSAT using different techniques

Many MaxSAT approaches



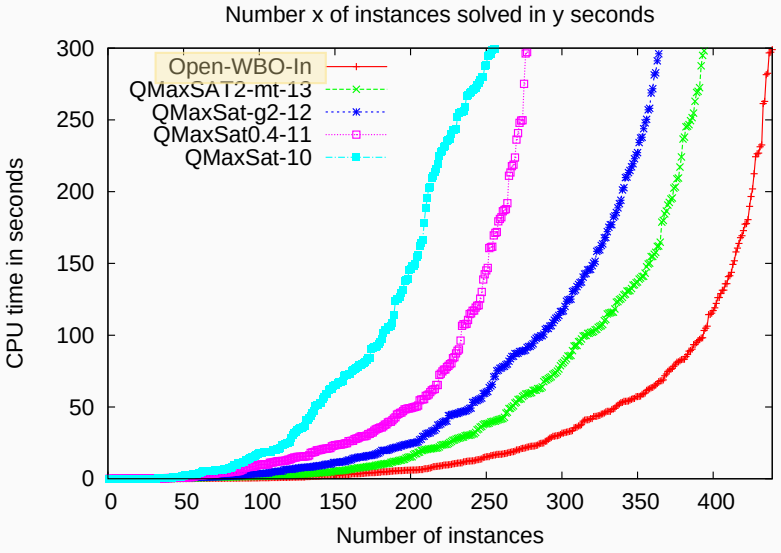
Many MaxSAT approaches



- For practical (**industrial**) instances: **core-guided** & **iterative MHS** approaches are the most effective

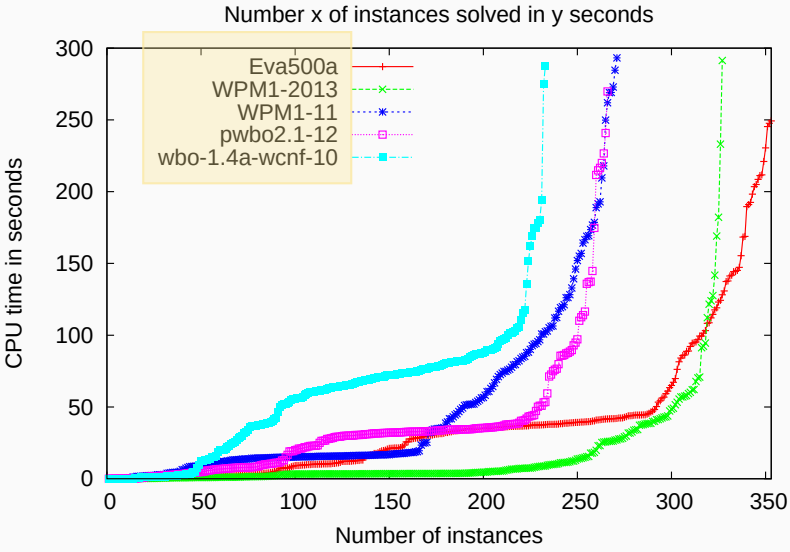
[MaxSAT14]

Core-guided solver performance – partial



Source: [MaxSAT 2014 organizers]

Core-guided solver performance – weighted partial



Source: [MaxSAT 2014 organizers]

Basic MaxSAT with iterative SAT solving

$$x_6 \vee x_2 \qquad \neg x_6 \vee x_2 \qquad \neg \neg x_2 \vee x_1 \qquad \neg x_1$$

$$\neg x_6 \vee x_8 \qquad x_6 \vee \neg x_8 \qquad x_2 \vee x_4 \qquad \neg x_4 \vee x_5$$

$$x_7 \vee x_5 \qquad \neg x_7 \vee x_5 \qquad \neg x_5 \vee x_3 \qquad \neg x_3$$

Example CNF formula

Basic MaxSAT with iterative SAT solving

$$x_6 \vee x_2 \vee r_1 \qquad \neg x_6 \vee x_2 \vee r_2 \qquad \neg x_2 \vee x_1 \vee r_3 \qquad \neg x_1 \vee r_4$$

$$\neg x_6 \vee x_8 \vee r_5 \qquad x_6 \vee \neg x_8 \vee r_6 \qquad x_2 \vee x_4 \vee r_7 \qquad \neg x_4 \vee x_5 \vee r_8$$

$$x_7 \vee x_5 \vee r_9 \qquad \neg x_7 \vee x_5 \vee r_{10} \qquad \neg x_5 \vee x_3 \vee r_{11} \qquad \neg x_3 \vee r_{12}$$

$$\sum_{i=1}^{12} r_i \leq 12$$

Relax all clauses; Set $UB = 12 + 1$

Basic MaxSAT with iterative SAT solving

$x_6 \vee x_2 \vee r_1$ $\quad \neg x_6 \vee x_2 \vee r_2$ $\quad \neg x_2 \vee x_1 \vee r_3$ $\quad \neg x_1 \vee r_4$

$\neg x_6 \vee x_8 \vee r_5$ $\quad x_6 \vee \neg x_8 \vee r_6$ $\quad x_2 \vee x_4 \vee r_7$ $\quad \neg x_4 \vee x_5 \vee r_8$

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$\sum_{i=1}^{12} r_i \leq 12$

Formula is SAT; E.g. all $x_i = 0$ and $r_1 = r_7 = r_9 = 1$ (i.e. cost $= 3$)

Basic MaxSAT with iterative SAT solving

$$x_6 \vee x_2 \vee r_1 \qquad \neg x_6 \vee x_2 \vee r_2 \qquad \neg x_2 \vee x_1 \vee r_3 \qquad \neg x_1 \vee r_4$$

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$$\sum_{i=1}^{12} r_i \leq 2$$

Refine $UB = 3$

Basic MaxSAT with iterative SAT solving

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$$\sum_{i=1}^{12} r_i \leq 2$$

Formula is SAT; E.g. $x_1 = x_2 = 1$; $x_3 = \ldots = x_8 = 0$ and $r_4 = r_9 = 1$ (i.e. cost $= 2$)

Basic MaxSAT with iterative SAT solving

$$x_6 \vee x_2 \vee r_1 \qquad \neg x_6 \vee x_2 \vee r_2 \qquad \neg x_2 \vee x_1 \vee r_3 \qquad \neg x_1 \vee r_4$$

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$$\sum_{i=1}^{12} r_i \leq 1$$

Refine $UB = 2$

Basic MaxSAT with iterative SAT solving

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$$\sum_{i=1}^{12} r_i \leq 1$$

Formula is UNSAT; terminate

Basic MaxSAT with iterative SAT solving

$$x_6 \vee x_2 \vee r_1 \qquad \neg x_6 \vee x_2 \vee r_2 \qquad \neg x_2 \vee x_1 \vee r_3 \qquad \neg x_1 \vee r_4$$

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$$\sum_{i=1}^{12} r_i \leq 1$$

MaxSAT solution is last satisfied UB: $UB = 2$

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$\sum_{i=1}^{12} r_i \leq 1$

MaxSAT solution is last satisfied UB: $UB = 2$

AtMost$k$/PB constraints over **all** relaxation variables

All (possibly many) soft clauses relaxed

MSU3 core-guided algorithm

$$x_6 \vee x_2 \qquad \neg x_6 \vee x_2 \qquad \neg x_2 \vee x_1 \qquad \neg x_1$$

$$\neg x_6 \vee x_8 \qquad x_6 \vee \neg x_8 \qquad x_2 \vee x_4 \qquad \neg x_4 \vee x_5$$

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Example CNF formula

MSU3 core-guided algorithm

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$x_7 \vee x_5$ $\quad$ $\neg x_7 \vee x_5$ $\quad$ $\neg x_5 \vee x_3$ $\quad$ $\neg x_3$

Formula is UNSAT; OPT $\leq |\varphi| - 1$; Get unsat core

MSU3 core-guided algorithm

$x_6 \vee x_2$ $\quad \neg x_6 \vee x_2$ $\quad \neg x_2 \vee x_1 \vee r_1$ $\quad \neg x_1 \vee r_2$

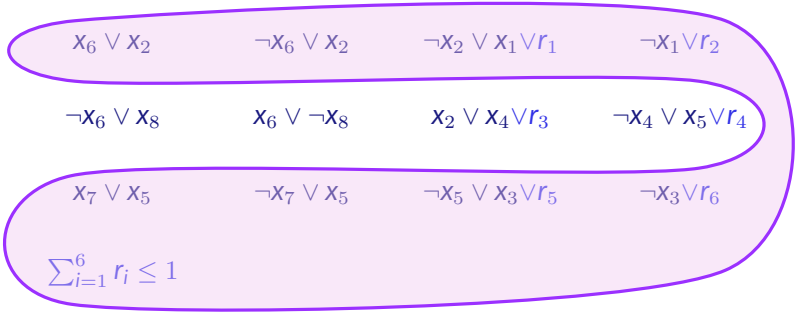
$\neg x_6 \vee x_8$ $\quad x_6 \vee \neg x_8$ $\quad x_2 \vee x_4 \vee r_3$ $\quad \neg x_4 \vee x_5 \vee r_4$

$x_7 \vee x_5$ $\quad \neg x_7 \vee x_5$ $\quad \neg x_5 \vee x_3 \vee r_5$ $\quad \neg x_3 \vee r_6$

$\sum_{i=1}^{6} r_i \leq 1$

Add relaxation variables and AtMost$k$, $k = 1$, constraint

MSU3 core-guided algorithm



Formula is (again) UNSAT; OPT $\leq |\varphi| - 2$; Get unsat core

MSU3 core-guided algorithm

$x_6 \vee x_2 \vee r_7$ $\quad \neg x_6 \vee x_2 \vee r_8$ $\quad \neg x_2 \vee x_1 \vee r_1$ $\quad \neg x_1 \vee r_2$

$\neg x_6 \vee x_8$ $\quad x_6 \vee \neg x_8$ $\quad x_2 \vee x_4 \vee r_3$ $\quad \neg x_4 \vee x_5 \vee r_4$

$x_7 \vee x_5 \vee r_9$ $\quad \neg x_7 \vee x_5 \vee r_{10}$ $\quad \neg x_5 \vee x_3 \vee r_5$ $\quad \neg x_3 \vee r_6$

$\sum_{i=1}^{10} r_i \leq 2$

Add new relaxation variables and update AtMost*k*, k=2, constraint

MSU3 core-guided algorithm

$x_6 \vee x_2 \vee r_7$ $\quad \neg x_6 \vee x_2 \vee r_8$ $\quad \neg x_2 \vee x_1 \vee r_1$ $\quad \neg x_1 \vee r_2$

$\neg x_6 \vee x_8$ $\quad x_6 \vee \neg x_8$ $\quad x_2 \vee x_4 \vee r_3$ $\quad \neg x_4 \vee x_5 \vee r_4$

$x_7 \vee x_5 \vee r_9$ $\quad \neg x_7 \vee x_5 \vee r_{10}$ $\quad \neg x_5 \vee x_3 \vee r_5$ $\quad \neg x_3 \vee r_6$

$\sum_{i=1}^{10} r_i \leq 2$

Instance is now SAT

MSU3 core-guided algorithm

$x_6 \vee x_2 \vee r_7$ $\quad \neg x_6 \vee x_2 \vee r_8$ $\quad \neg x_2 \vee x_1 \vee r_1$ $\quad \neg x_1 \vee r_2$

$\neg x_6 \vee x_8$ $\quad x_6 \vee \neg x_8$ $\quad x_2 \vee x_4 \vee r_3$ $\quad \neg x_4 \vee x_5 \vee r_4$

$x_7 \vee x_5 \vee r_9$ $\quad \neg x_7 \vee x_5 \vee r_{10}$ $\quad \neg x_5 \vee x_3 \vee r_5$ $\quad \neg x_3 \vee r_6$

$\sum_{i=1}^{10} r_i \leq 2$

MaxSAT solution is $|\varphi| - \mathcal{I} = 12 - 2 = 10$

MSU3 core-guided algorithm

$x_6 \vee x_2 \vee r_7$ $\quad \neg x_6 \vee x_2 \vee r_8$ $\quad \neg x_2 \vee x_1 \vee r_1$ $\quad \neg x_1 \vee r_2$

$\neg x_6 \vee x_8$ $\quad x_6 \vee \neg x_8$ $\quad x_2 \vee x_4 \vee r_3$ $\quad \neg x_4 \vee x_5 \vee r_4$

$x_7 \vee x_5 \vee r_9$ $\quad \neg x_7 \vee x_5 \vee r_{10}$ $\quad \neg x_5 \vee x_3 \vee r_5$ $\quad \neg x_3 \vee r_6$

$\sum_{i=1}^{10} r_i \leq 2$

MaxSAT solution is $|\varphi| - \mathcal{I} = 12 - 2 = 10$

AtMost$k$/PB constraints used

Relaxed soft clauses become **hard**

MSU3 core-guided algorithm

$x_6 \vee x_2 \vee r_7$ $\quad \neg x_6 \vee x_2 \vee r_8$ $\quad \neg x_2 \vee x_1 \vee r_1$ $\quad \neg x_1 \vee r_2$

$\neg x_6 \vee x_8$ $\quad x_6 \vee \neg x_8$ $\quad x_2 \vee x_4 \vee r_3$ $\quad \neg x_4 \vee x_5 \vee r_4$

$x_7 \vee x_5 \vee r_9$ $\quad \neg x_7 \vee x_5 \vee r_{10}$ $\quad \neg x_5 \vee x_3 \vee r_5$ $\quad \neg x_3 \vee r_6$

$\sum_{i=1}^{10} r_i \leq 2$

MaxSAT solution is $|\rho| - \mathcal{I} = 12 - 2 = 10$

AtMost$k$/PB constraints used

Some clauses not relaxed

Relaxed soft clauses become **hard**

MHS approach for MaxSAT

$$c_1 = x_6 \vee x_2 \qquad c_2 = \neg x_6 \vee x_2 \qquad c_3 = \neg x_2 \vee x_1 \qquad c_4 = \neg x_1$$

$$c_5 = \neg x_6 \vee x_8 \qquad c_6 = x_6 \vee \neg x_8 \qquad c_7 = x_2 \vee x_4 \qquad c_8 = \neg x_4 \vee x_5$$

$$c_9 = x_7 \vee x_5 \qquad c_{10} = \neg x_7 \vee x_5 \qquad c_{11} = \neg x_5 \vee x_3 \qquad c_{12} = \neg x_3$$

$$\mathcal{K} = \emptyset$$

- Find MHS of $\mathcal{K}$:

MHS approach for MaxSAT

$$c_1 = x_6 \vee x_2 \qquad c_2 = \neg x_6 \vee x_2 \qquad c_3 = \neg x_2 \vee x_1 \qquad c_4 = \neg x_1$$

$$c_5 = \neg x_6 \vee x_8 \qquad c_6 = x_6 \vee \neg x_8 \qquad c_7 = x_2 \vee x_4 \qquad c_8 = \neg x_4 \vee x_5$$

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$$\mathcal{K} = \emptyset$$

- Find MHS of $\mathcal{K}$: $\emptyset$

MHS approach for MaxSAT

$$c_1 = x_6 \vee x_2 \qquad c_2 = \neg x_6 \vee x_2 \qquad c_3 = \neg x_2 \vee x_1 \qquad c_4 = \neg x_1$$

$$c_5 = \neg x_6 \vee x_8 \qquad c_6 = x_6 \vee \neg x_8 \qquad c_7 = x_2 \vee x_4 \qquad c_8 = \neg x_4 \vee x_5$$

$$c_9 = x_7 \vee x_5 \qquad c_{10} = \neg x_7 \vee x_5 \qquad c_{11} = \neg x_5 \vee x_3 \qquad c_{12} = \neg x_3$$

$$\mathcal{K} = \emptyset$$

- Find MHS of $\mathcal{K}$: $\emptyset$
- SAT($\mathcal{F} \setminus \emptyset$)?

MHS approach for MaxSAT

$$c_1 = x_6 \vee x_2 \qquad c_2 = \neg x_6 \vee x_2 \qquad c_3 = \neg x_2 \vee x_1 \qquad c_4 = \neg x_1$$

$$c_5 = \neg x_6 \vee x_8 \qquad c_6 = x_6 \vee \neg x_8 \qquad c_7 = x_2 \vee x_4 \qquad c_8 = \neg x_4 \vee x_5$$

$$c_9 = x_7 \vee x_5 \qquad c_{10} = \neg x_7 \vee x_5 \qquad c_{11} = \neg x_5 \vee x_3 \qquad c_{12} = \neg x_3$$

$$\mathcal{K} = \emptyset$$

- Find MHS of $\mathcal{K}$: $\emptyset$
- SAT($\mathcal{F} \setminus \emptyset$)? No

MHS approach for MaxSAT

$$c_1 = x_6 \vee x_2 \qquad c_2 = \neg x_6 \vee x_2 \qquad c_3 = \neg x_2 \vee x_1 \qquad c_4 = \neg x_1$$

$$c_5 = \neg x_6 \vee x_8 \qquad c_6 = x_6 \vee \neg x_8 \qquad c_7 = x_2 \vee x_4 \qquad c_8 = \neg x_4 \vee x_5$$

$$c_9 = x_7 \vee x_5 \qquad c_{10} = \neg x_7 \vee x_5 \qquad c_{11} = \neg x_5 \vee x_3 \qquad c_{12} = \neg x_3$$

$$\mathcal{K} = \emptyset$$

- Find MHS of $\mathcal{K}$: $\emptyset$
- SAT($\mathcal{F} \setminus \emptyset$)? No
- Core of $\mathcal{F}$: $\{c_1, c_2, c_3, c_4\}$

MHS approach for MaxSAT

$$c_1 = x_6 \vee x_2 \qquad c_2 = \neg x_6 \vee x_2 \qquad c_3 = \neg x_2 \vee x_1 \qquad c_4 = \neg x_1$$

$$c_5 = \neg x_6 \vee x_8 \qquad c_6 = x_6 \vee \neg x_8 \qquad c_7 = x_2 \vee x_4 \qquad c_8 = \neg x_4 \vee x_5$$

$$c_9 = x_7 \vee x_5 \qquad c_{10} = \neg x_7 \vee x_5 \qquad c_{11} = \neg x_5 \vee x_3 \qquad c_{12} = \neg x_3$$

$$\mathcal{K} = \{\{c_1, c_2, c_3, c_4\}\}$$

- Find MHS of $\mathcal{K}$: $\emptyset$
- SAT($\mathcal{F} \setminus \emptyset$)? No
- Core of $\mathcal{F}$: $\{c_1, c_2, c_3, c_4\}$. Update $\mathcal{K}$

MHS approach for MaxSAT

$$c_1 = x_6 \vee x_2 \qquad c_2 = \neg x_6 \vee x_2 \qquad c_3 = \neg x_2 \vee x_1 \qquad c_4 = \neg x_1$$

$$c_5 = \neg x_6 \vee x_8 \qquad c_6 = x_6 \vee \neg x_8 \qquad c_7 = x_2 \vee x_4 \qquad c_8 = \neg x_4 \vee x_5$$

$$c_9 = x_7 \vee x_5 \qquad c_{10} = \neg x_7 \vee x_5 \qquad c_{11} = \neg x_5 \vee x_3 \qquad c_{12} = \neg x_3$$

$$\mathcal{K} = \{\{c_1, c_2, c_3, c_4\}\}$$

- Find MHS of $\mathcal{K}$:

MHS approach for MaxSAT

$$c_1 = x_6 \vee x_2 \qquad c_2 = \neg x_6 \vee x_2 \qquad c_3 = \neg x_2 \vee x_1 \qquad c_4 = \neg x_1$$

$$c_5 = \neg x_6 \vee x_8 \qquad c_6 = x_6 \vee \neg x_8 \qquad c_7 = x_2 \vee x_4 \qquad c_8 = \neg x_4 \vee x_5$$

$$c_9 = x_7 \vee x_5 \qquad c_{10} = \neg x_7 \vee x_5 \qquad c_{11} = \neg x_5 \vee x_3 \qquad c_{12} = \neg x_3$$

$$\mathcal{K} = \{\{c_1, c_2, c_3, c_4\}\}$$

- Find MHS of $\mathcal{K}$: E.g. $\{c_1\}$

MHS approach for MaxSAT

$$c_1 = x_6 \vee x_2 \qquad c_2 = \neg x_6 \vee x_2 \qquad c_3 = \neg x_2 \vee x_1 \qquad c_4 = \neg x_1$$

$$c_5 = \neg x_6 \vee x_8 \qquad c_6 = x_6 \vee \neg x_8 \qquad c_7 = x_2 \vee x_4 \qquad c_8 = \neg x_4 \vee x_5$$

$$c_9 = x_7 \vee x_5 \qquad c_{10} = \neg x_7 \vee x_5 \qquad c_{11} = \neg x_5 \vee x_3 \qquad c_{12} = \neg x_3$$

$$\mathcal{K} = \{\{c_1, c_2, c_3, c_4\}\}$$

- Find MHS of $\mathcal{K}$: E.g. $\{c_1\}$
- SAT($\mathcal{F} \setminus \{c_1\}$)?

MHS approach for MaxSAT

$$c_1 = x_6 \vee x_2 \qquad c_2 = \neg x_6 \vee x_2 \qquad c_3 = \neg x_2 \vee x_1 \qquad c_4 = \neg x_1$$

$$c_5 = \neg x_6 \vee x_8 \qquad c_6 = x_6 \vee \neg x_8 \qquad c_7 = x_2 \vee x_4 \qquad c_8 = \neg x_4 \vee x_5$$

$$c_9 = x_7 \vee x_5 \qquad c_{10} = \neg x_7 \vee x_5 \qquad c_{11} = \neg x_5 \vee x_3 \qquad c_{12} = \neg x_3$$

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- Find MHS of $\mathcal{K}$: E.g. $\{c_1\}$
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MHS approach for MaxSAT

$$c_1 = x_6 \vee x_2 \qquad c_2 = \neg x_6 \vee x_2 \qquad c_3 = \neg x_2 \vee x_1 \qquad c_4 = \neg x_1$$

$$c_5 = \neg x_6 \vee x_8 \qquad c_6 = x_6 \vee \neg x_8 \qquad c_7 = x_2 \vee x_4 \qquad c_8 = \neg x_4 \vee x_5$$

$$c_9 = x_7 \vee x_5 \qquad c_{10} = \neg x_7 \vee x_5 \qquad c_{11} = \neg x_5 \vee x_3 \qquad c_{12} = \neg x_3$$

$$\mathcal{K} = \{\{c_1, c_2, c_3, c_4\}\}$$

- Find MHS of $\mathcal{K}$: E.g. $\{c_1\}$
- SAT($\mathcal{F} \setminus \{c_1\}$)? No
- Core of $\mathcal{F}$: $\{c_9, c_{10}, c_{11}, c_{12}\}$

MHS approach for MaxSAT

$$c_1 = x_6 \vee x_2 \qquad c_2 = \neg x_6 \vee x_2 \qquad c_3 = \neg x_2 \vee x_1 \qquad c_4 = \neg x_1$$

$$c_5 = \neg x_6 \vee x_8 \qquad c_6 = x_6 \vee \neg x_8 \qquad c_7 = x_2 \vee x_4 \qquad c_8 = \neg x_4 \vee x_5$$

$$c_9 = x_7 \vee x_5 \qquad c_{10} = \neg x_7 \vee x_5 \qquad c_{11} = \neg x_5 \vee x_3 \qquad c_{12} = \neg x_3$$

$$\mathcal{K} = \{\{c_1, c_2, c_3, c_4\}, \{c_9, c_{10}, c_{11}, c_{12}\}\}$$

- Find MHS of $\mathcal{K}$: E.g. $\{c_1\}$
- SAT($\mathcal{F} \setminus \{c_1\}$)? No
- Core of $\mathcal{F}$: $\{c_9, c_{10}, c_{11}, c_{12}\}$. Update $\mathcal{K}$

MHS approach for MaxSAT

$$c_1 = x_6 \vee x_2 \qquad c_2 = \neg x_6 \vee x_2 \qquad c_3 = \neg x_2 \vee x_1 \qquad c_4 = \neg x_1$$

$$c_5 = \neg x_6 \vee x_8 \qquad c_6 = x_6 \vee \neg x_8 \qquad c_7 = x_2 \vee x_4 \qquad c_8 = \neg x_4 \vee x_5$$

$$c_9 = x_7 \vee x_5 \qquad c_{10} = \neg x_7 \vee x_5 \qquad c_{11} = \neg x_5 \vee x_3 \qquad c_{12} = \neg x_3$$

$$\mathcal{K} = \{\{c_1, c_2, c_3, c_4\}, \{c_9, c_{10}, c_{11}, c_{12}\}\}$$

- Find MHS of $\mathcal{K}$:

MHS approach for MaxSAT

$$c_1 = x_6 \vee x_2 \qquad c_2 = \neg x_6 \vee x_2 \qquad c_3 = \neg x_2 \vee x_1 \qquad c_4 = \neg x_1$$

$$c_5 = \neg x_6 \vee x_8 \qquad c_6 = x_6 \vee \neg x_8 \qquad c_7 = x_2 \vee x_4 \qquad c_8 = \neg x_4 \vee x_5$$

$$c_9 = x_7 \vee x_5 \qquad c_{10} = \neg x_7 \vee x_5 \qquad c_{11} = \neg x_5 \vee x_3 \qquad c_{12} = \neg x_3$$

$$\mathcal{K} = \{\{c_1, c_2, c_3, c_4\}, \{c_9, c_{10}, c_{11}, c_{12}\}\}$$

- Find MHS of $\mathcal{K}$: E.g. $\{c_1, c_9\}$

MHS approach for MaxSAT

$$c_1 = x_6 \vee x_2 \qquad c_2 = \neg x_6 \vee x_2 \qquad c_3 = \neg x_2 \vee x_1 \qquad c_4 = \neg x_1$$

$$c_5 = \neg x_6 \vee x_8 \qquad c_6 = x_6 \vee \neg x_8 \qquad c_7 = x_2 \vee x_4 \qquad c_8 = \neg x_4 \vee x_5$$

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$$\mathcal{K} = \{\{c_1, c_2, c_3, c_4\}, \{c_9, c_{10}, c_{11}, c_{12}\}\}$$

- Find MHS of $\mathcal{K}$: E.g. $\{c_1, c_9\}$
- SAT($\mathcal{F} \setminus \{c_1, c_9\}$)?

MHS approach for MaxSAT

$$c_1 = x_6 \vee x_2 \qquad c_2 = \neg x_6 \vee x_2 \qquad c_3 = \neg x_2 \vee x_1 \qquad c_4 = \neg x_1$$

$$c_5 = \neg x_6 \vee x_8 \qquad c_6 = x_6 \vee \neg x_8 \qquad c_7 = x_2 \vee x_4 \qquad c_8 = \neg x_4 \vee x_5$$

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$$\mathcal{K} = \{\{c_1, c_2, c_3, c_4\}, \{c_9, c_{10}, c_{11}, c_{12}\}\}$$

- Find MHS of $\mathcal{K}$: E.g. $\{c_1, c_9\}$
- SAT($\mathcal{F} \setminus \{c_1, c_9\}$)? No

MHS approach for MaxSAT

$$c_1 = x_6 \vee x_2 \qquad c_2 = \neg x_6 \vee x_2 \qquad c_3 = \neg x_2 \vee x_1 \qquad c_4 = \neg x_1$$

$$c_5 = \neg x_6 \vee x_8 \qquad c_6 = x_6 \vee \neg x_8 \qquad c_7 = x_2 \vee x_4 \qquad c_8 = \neg x_4 \vee x_5$$

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$$\mathcal{K} = \{\{c_1, c_2, c_3, c_4\}, \{c_9, c_{10}, c_{11}, c_{12}\}\}$$

- Find MHS of $\mathcal{K}$: E.g. $\{c_1, c_9\}$
- SAT($\mathcal{F} \setminus \{c_1, c_9\}$)? No
- Core of $\mathcal{F}$: $\{c_3, c_4, c_7, c_8, c_{11}, c_{12}\}$

MHS approach for MaxSAT

$$c_1 = x_6 \vee x_2 \qquad c_2 = \neg x_6 \vee x_2 \qquad c_3 = \neg x_2 \vee x_1 \qquad c_4 = \neg x_1$$

$$c_5 = \neg x_6 \vee x_8 \qquad c_6 = x_6 \vee \neg x_8 \qquad c_7 = x_2 \vee x_4 \qquad c_8 = \neg x_4 \vee x_5$$

$$c_9 = x_7 \vee x_5 \qquad c_{10} = \neg x_7 \vee x_5 \qquad c_{11} = \neg x_5 \vee x_3 \qquad c_{12} = \neg x_3$$

$$\mathcal{K} = \{\{c_1, c_2, c_3, c_4\}, \{c_9, c_{10}, c_{11}, c_{12}\}, \{c_3, c_4, c_7, c_8, c_{11}, c_{12}\}\}$$

- Find MHS of $\mathcal{K}$: E.g. $\{c_1, c_9\}$
- SAT($\mathcal{F} \setminus \{c_1, c_9\}$)? No
- Core of $\mathcal{F}$: $\{c_3, c_4, c_7, c_8, c_{11}, c_{12}\}$. Update $\mathcal{K}$

MHS approach for MaxSAT

$$c_1 = x_6 \vee x_2 \qquad c_2 = \neg x_6 \vee x_2 \qquad c_3 = \neg x_2 \vee x_1 \qquad c_4 = \neg x_1$$

$$c_5 = \neg x_6 \vee x_8 \qquad c_6 = x_6 \vee \neg x_8 \qquad c_7 = x_2 \vee x_4 \qquad c_8 = \neg x_4 \vee x_5$$

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$$\mathcal{K} = \{\{c_1, c_2, c_3, c_4\}, \{c_9, c_{10}, c_{11}, c_{12}\}, \{c_3, c_4, c_7, c_8, c_{11}, c_{12}\}\}$$

- Find MHS of $\mathcal{K}$:

MHS approach for MaxSAT

$$c_1 = x_6 \vee x_2 \qquad c_2 = \neg x_6 \vee x_2 \qquad c_3 = \neg x_2 \vee x_1 \qquad c_4 = \neg x_1$$

$$c_5 = \neg x_6 \vee x_8 \qquad c_6 = x_6 \vee \neg x_8 \qquad c_7 = x_2 \vee x_4 \qquad c_8 = \neg x_4 \vee x_5$$

$$c_9 = x_7 \vee x_5 \qquad c_{10} = \neg x_7 \vee x_5 \qquad c_{11} = \neg x_5 \vee x_3 \qquad c_{12} = \neg x_3$$

$$\mathcal{K} = \{\{c_1, c_2, c_3, c_4\}, \{c_9, c_{10}, c_{11}, c_{12}\}, \{c_3, c_4, c_7, c_8, c_{11}, c_{12}\}\}$$

- Find MHS of $\mathcal{K}$: E.g. $\{c_4, c_9\}$

MHS approach for MaxSAT

$$c_1 = x_6 \vee x_2 \qquad c_2 = \neg x_6 \vee x_2 \qquad c_3 = \neg x_2 \vee x_1 \qquad c_4 = \neg x_1$$

$$c_5 = \neg x_6 \vee x_8 \qquad c_6 = x_6 \vee \neg x_8 \qquad c_7 = x_2 \vee x_4 \qquad c_8 = \neg x_4 \vee x_5$$

$$c_9 = x_7 \vee x_5 \qquad c_{10} = \neg x_7 \vee x_5 \qquad c_{11} = \neg x_5 \vee x_3 \qquad c_{12} = \neg x_3$$

$$\mathcal{K} = \{\{c_1, c_2, c_3, c_4\}, \{c_9, c_{10}, c_{11}, c_{12}\}, \{c_3, c_4, c_7, c_8, c_{11}, c_{12}\}\}$$

- Find MHS of $\mathcal{K}$: E.g. $\{c_4, c_9\}$
- SAT($\mathcal{F} \setminus \{c_4, c_9\}$)?

MHS approach for MaxSAT

$$c_1 = x_6 \vee x_2 \qquad c_2 = \neg x_6 \vee x_2 \qquad c_3 = \neg x_2 \vee x_1 \qquad c_4 = \neg x_1$$

$$c_5 = \neg x_6 \vee x_8 \qquad c_6 = x_6 \vee \neg x_8 \qquad c_7 = x_2 \vee x_4 \qquad c_8 = \neg x_4 \vee x_5$$

$$c_9 = x_7 \vee x_5 \qquad c_{10} = \neg x_7 \vee x_5 \qquad c_{11} = \neg x_5 \vee x_3 \qquad c_{12} = \neg x_3$$

$$\mathcal{K} = \{\{c_1, c_2, c_3, c_4\}, \{c_9, c_{10}, c_{11}, c_{12}\}, \{c_3, c_4, c_7, c_8, c_{11}, c_{12}\}\}$$

- Find MHS of $\mathcal{K}$: E.g. $\{c_4, c_9\}$
- SAT($\mathcal{F} \setminus \{c_4, c_9\}$)? Yes

MHS approach for MaxSAT

$$c_1 = x_6 \vee x_2 \qquad c_2 = \neg x_6 \vee x_2 \qquad c_3 = \neg x_2 \vee x_1 \qquad c_4 = \neg x_1$$

$$c_5 = \neg x_6 \vee x_8 \qquad c_6 = x_6 \vee \neg x_8 \qquad c_7 = x_2 \vee x_4 \qquad c_8 = \neg x_4 \vee x_5$$

$$c_9 = x_7 \vee x_5 \qquad c_{10} = \neg x_7 \vee x_5 \qquad c_{11} = \neg x_5 \vee x_3 \qquad c_{12} = \neg x_3$$

$$\mathcal{K} = \{\{c_1, c_2, c_3, c_4\}, \{c_9, c_{10}, c_{11}, c_{12}\}, \{c_3, c_4, c_7, c_8, c_{11}, c_{12}\}\}$$

- Find MHS of $\mathcal{K}$: E.g. $\{c_4, c_9\}$
- SAT($\mathcal{F} \setminus \{c_4, c_9\}$)? Yes
- Terminate & return 2

MaxSAT solving with SAT oracles – a sample

- A sample of recent algorithms:

Algorithm	# Oracle Queries	Reference
Linear search SU	Exponential***	[BP10]
Binary search	Linear*	[FM06]
FM/WMSU1/WPM1	Exponential**	[FM06, MP08, MMSP09, ABL09, ABGL12]
WPM2	Exponential**	[ABL10a, ABL13]
Bin-Core-Dis	Linear	[HMM11, MHM12]
Iterative MHS	Exponential	[DB11, DB13a, DB13b]

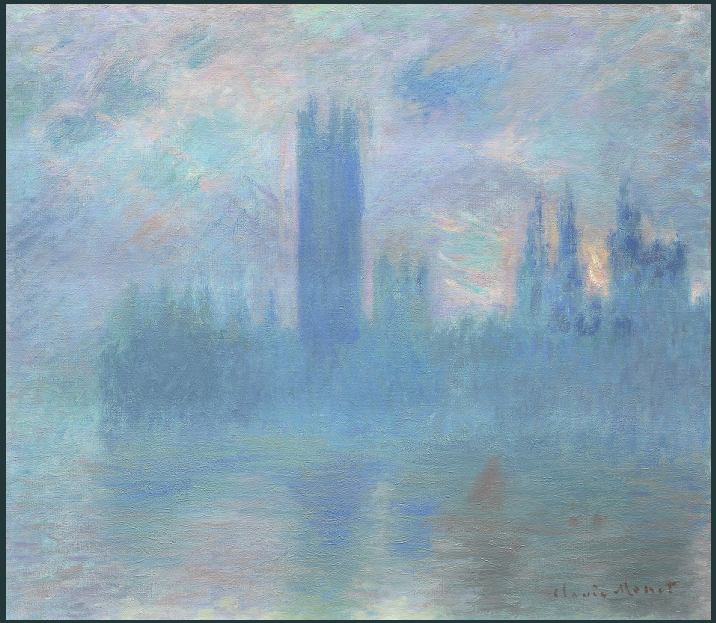
* $\mathcal{O}(\log m)$ queries with SAT oracle, for (partial) unweighted MaxSAT

** Weighted case; depends on computed cores

*** On # bits of problem instance (due to weights)

- But also additional recent work:
  - Progression [IMM+14]
  - Soft cardinality constraints (OLL) [MDM14, MIM14]
    - Recent implementation (RC2, using PySAT) won 2018 MaxSAT Evaluation
  - MaxSAT resolution [NB14]
  - ...

5 Sample of Applications



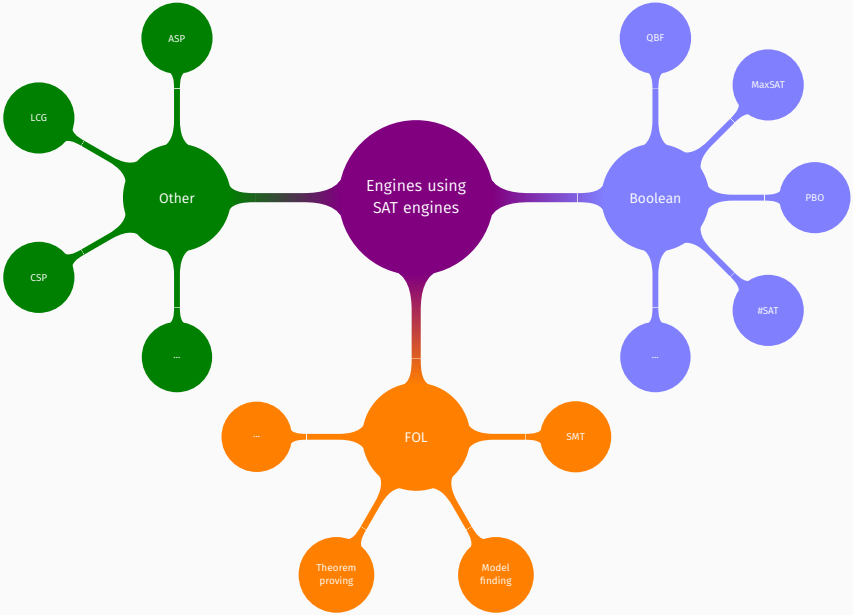
Flagship applications

- Bounded (& unbounded) model checking
- Automated planning
- Multi-agent path finding

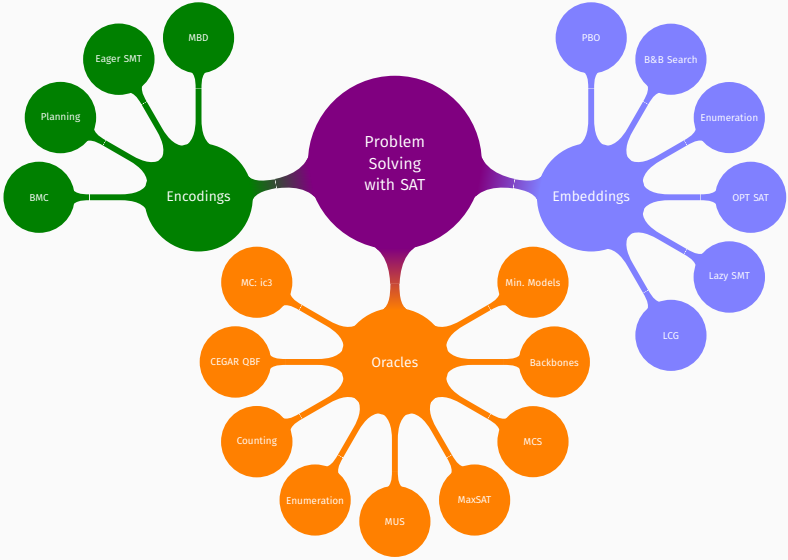
- Software model checking
- Package management
- Design debugging

- Haplotyping

CDCL SAT is the engines' engine



CDCL SAT is ubiquitous in problem solving



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  - **Abduction-based explanations for ML models** [INMS19]
    - Extraction of explanations for ***any*** ML model **on demand**

Package dependency

Package dependency

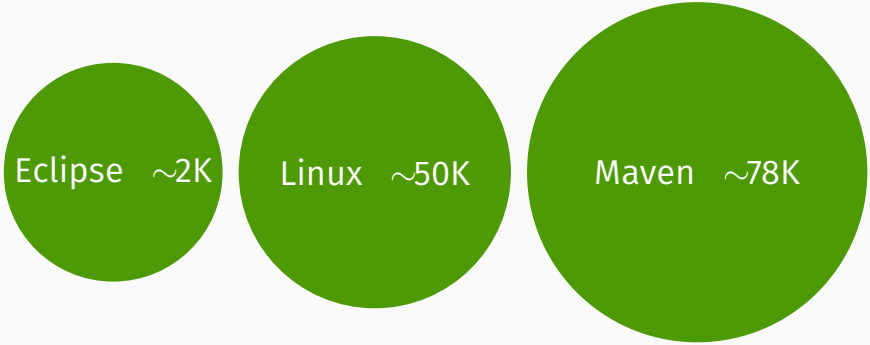


Figure 1: Number of packages in modern package management systems

Package dependency

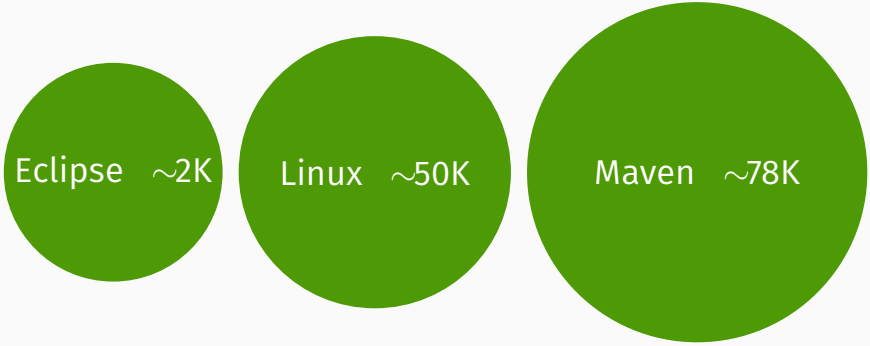
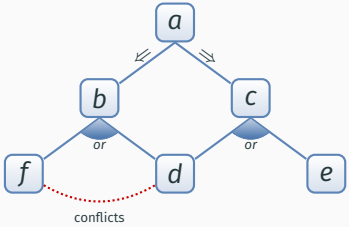


Figure 1: Number of packages in modern package management systems

can single package *P* **be installed** in repository *R*? — **NP-complete!**

PackUP example



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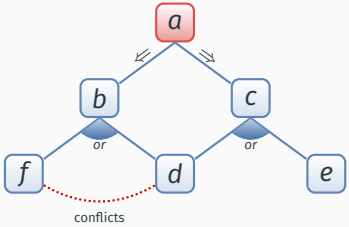

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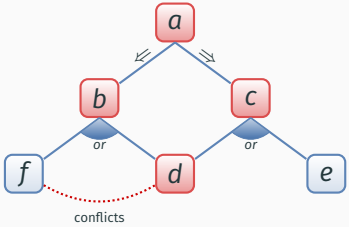

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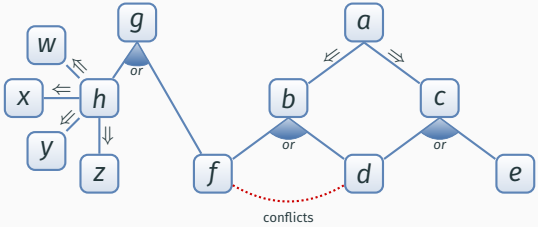

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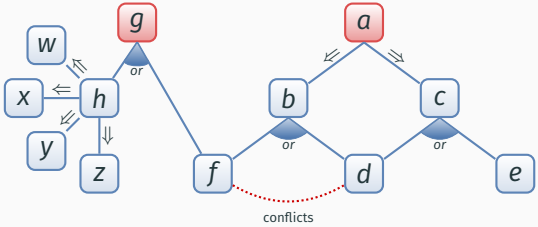

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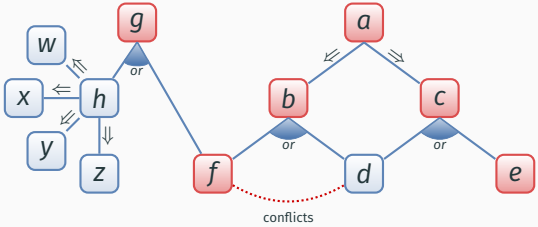
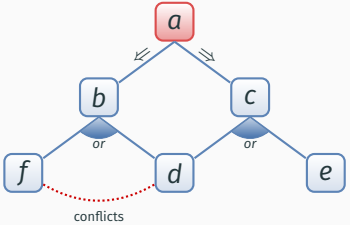


Encoding PackUP to SAT and MaxSAT



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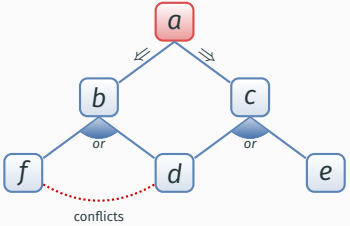

Find any solution — SAT

$$(\neg a \vee b) \wedge (\neg a \vee c) \wedge (\neg b \vee f \vee d) \wedge (\neg c \vee d \vee e) \wedge (\neg f \vee \neg d) \wedge (a)$$

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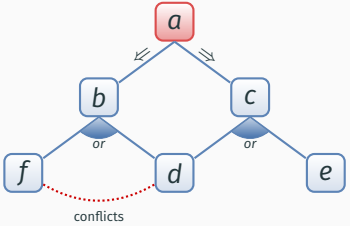

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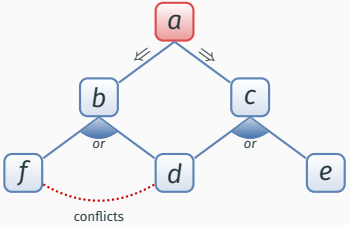

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Solving MaxClique with SAT

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- given a graph $G = (V, E)$, find a **largest complete subgraph**

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- **main goal** — **maximize** $\sum_{u \in V} x_u$
  - e.g. use MaxSAT

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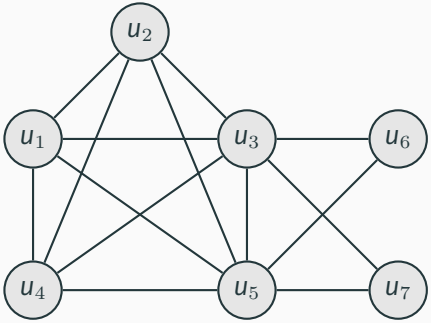

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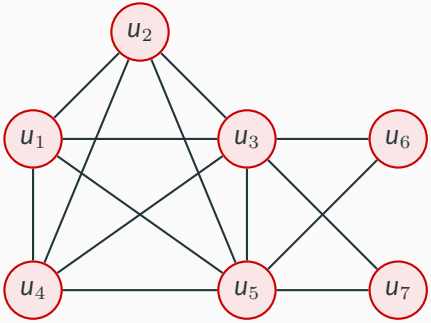

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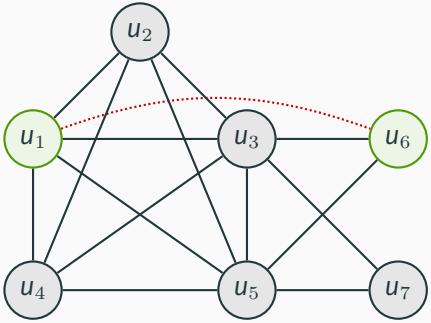

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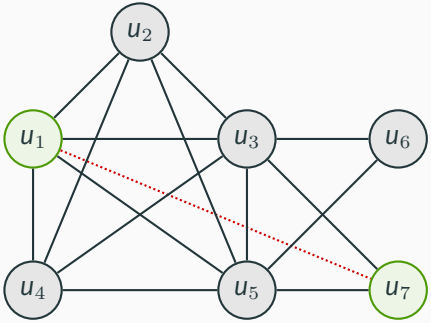

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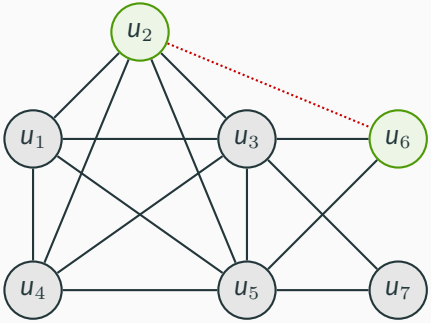

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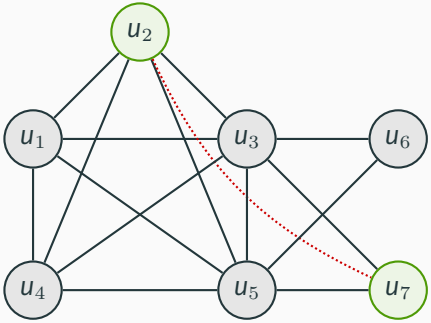

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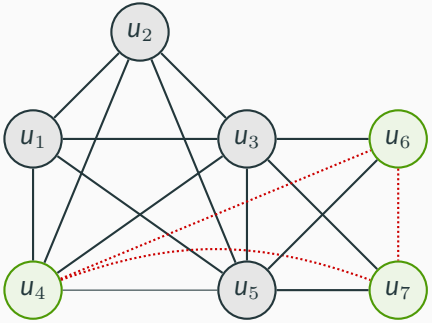

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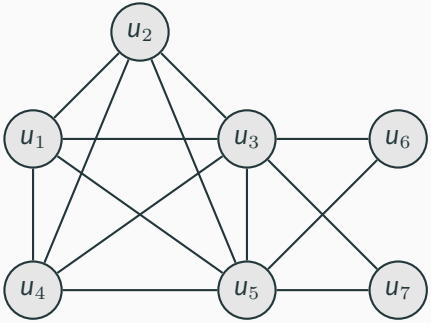

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solve $\mathcal{F}$ with MaxSAT !

But the size of E^C can be problematic...

Instance	$\|V\|$	$\|E\|$	$\|E\|^C$
comm-n10000	10000	10000	49995000
ca-AstroPh	18772	396160	175807218
ca-citeseer	227322	814136	25836945367
ca-coauthors-dblp	540488	15245731	146048663585
ca-CondMat	23133	186936	267392475
ca-dblp-2010	226415	716462	25631272858
ca-dblp-2012	317082	1049868	50269606035
ca-HepPh	12008	237010	71865026
ca-HepTh	9877	51971	48730532
ca-MathSciNet	332689	820644	55340331061
ia-email-EU	32430	54397	525814268
ia-reality-call	6809	9484	23175161
ia-retweet-pol	18470	61157	170518528
ia-wiki-Talk	92117	360767	4242456136
rt-pol	18470	61157	170518528
rt_barackobama	9631	9826	46373070
soc-epinions	63947	606512	2044034866
soc-gplus	23628	39242	279113764
tech-as-caida2007	26477	53383	350475620
tech-internet-as	40164	85123	806508407
tech-pgp	10680	24340	57012200
tech-WHOIS	7476	56943	27892083
web-arabic-2005	163598	1747269	13380487332
web-baidu-baike-related	415641	3284387	86375643874
web-it-2004	509338	7178413	129705675378
web-NotreDame	325729	1497134	53048356451
web-sk-2005	121422	334419	7371377334

But the size of E^C can be problematic...

Instance	$\|V\|$	$\|E\|$	$\|E\|^C$
comm-n10000	10000	10000	49995000
ca-AstroPh	18772	396160	175807218
ca-citeseer	227322	814136	25836945367
ca-coauthors-dblp	540488	15245731	146048663585
ca-CondMat	23133	186936	267392475
ca-dblp-2010	226415	716462	25631272858
ca-dblp-2012	317082	1049868	50269606035
ca-HepPh	12008	237010	71865026
ca-HepTh	9877	51971	48730532
ca-MathSciNet	332689	820644	55340331061
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$$|E^C| = \frac{|E| \times (|E|-1)}{2} - |E|$$

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$$|E^C| = \frac{|E| \times (|E|-1)}{2} - |E|$$

Unrealistic to model with SAT on sparse graphs

How to reduce the encoding size?

- **main hurdle:**
  - approaches based on $G^C = (V, E^C)$ **will not** scale...
  - and $G = (V, E)$ is **much smaller** than $G^C = (V, E^C)$

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- can we model MaxClique using **solely** G?

Another take at solving MaxClique with SAT

revisit the original **decision problem**:

given $G = (V, E)$, is there a clique of **size** K?

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Another take at solving MaxClique with SAT

revisit the original **decision problem**:
given $G = (V, E)$, is there a clique of **size** K?

1. one **must** pick exactly K vertices:

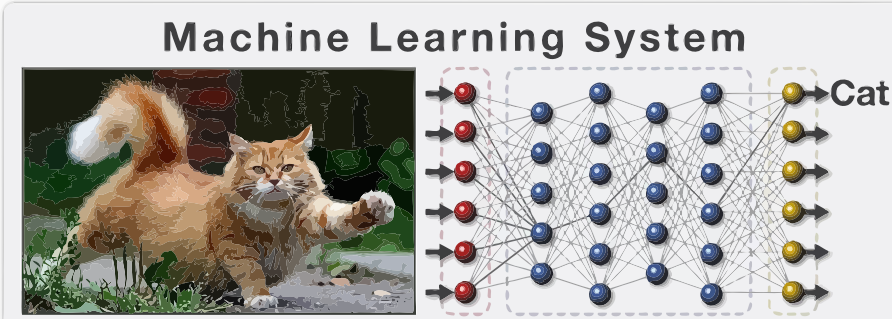
$$\sum_{u \in V} x_u = K$$

2. if a vertex $u \in V$ is picked (i.e. $x_u = 1$), **then** $K - 1$ of its neighbours **must** also be picked:

$$x_u \rightarrow \left(\sum_{v \in \mathrm{Adj}(u)} x_v = K - 1 \right)$$

eXplainable AI (XAI)

What is eXplainable AI (XAI)?



interpretable ML models

(decision trees, lists, sets)

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(decision trees, lists, sets)

explanation of ML models **“on the fly”**

Interpretable ML models

Example	Lecture	Concert	Expo	Shop	*Hike?*
e_1	1	0	1	0	0
e_2	1	0	0	1	0
e_3	0	0	1	0	1
e_4	1	1	0	0	0
e_5	0	0	0	1	1
e_6	1	1	1	1	0
e_7	0	1	1	0	0
e_8	0	0	1	1	1

(a) When should we **hike** and when **not**?

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(a) When should we **hike** and when **not**?

if ¬**Lecture** **and** ¬**Concert** **then** *Hike*
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(b) Example decision set

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if ¬**Lecture** **and** ¬**Concert** **then** *Hike*
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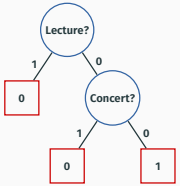
if **Concert** **then** ¬*Hike*

(b) Example decision set



(c) Example decision tree

[IPNM18, NIPM18]

can be **encoded** into SAT

a series of SAT oracle calls

Smallest decision trees – encoding sizes in bytes

[NIPM18]

Model	Weather	Mouse	Cancer	Car	Income
CP'09*	27K	3.5M	92G	842M	354G

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Model	Weather	Mouse	Cancer	Car	Income
CP'09*	27K	3.5M	92G	842M	354G
IJCAI'18	190K	1.2M	5.2M	4.1M	1.2G

Post-hoc explanations

heuristic approaches exist

(e.g. **LIME** or **Anchor**)

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- **local** explanations

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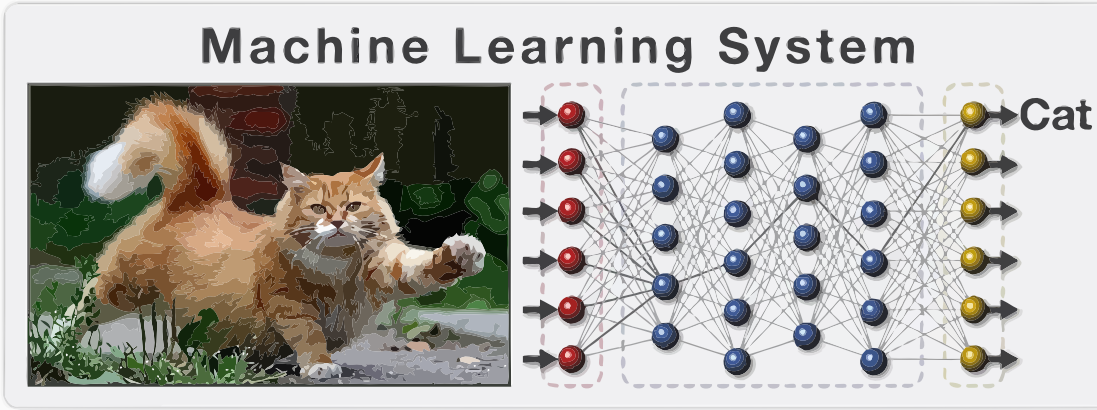
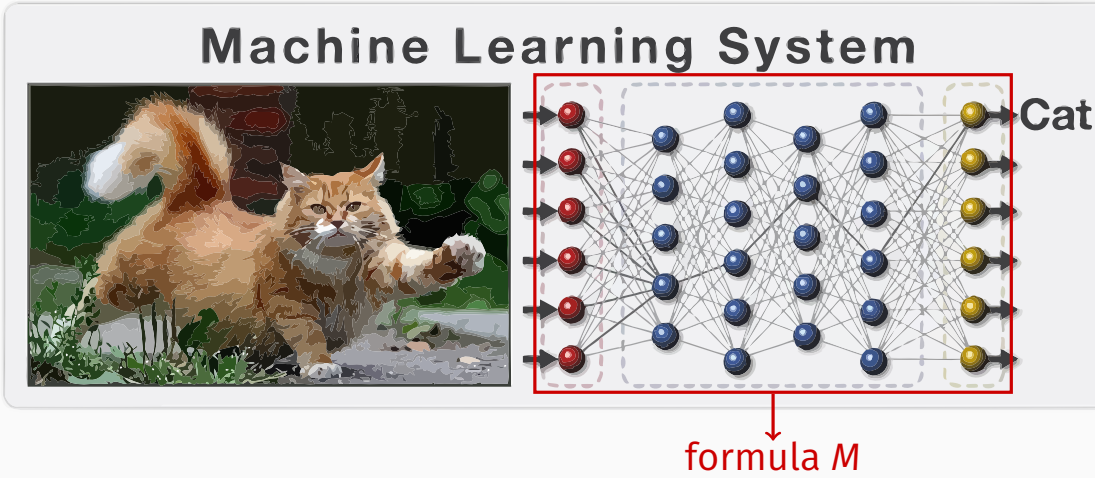
(un-)reliable?

From ML model to logic



From ML model to logic



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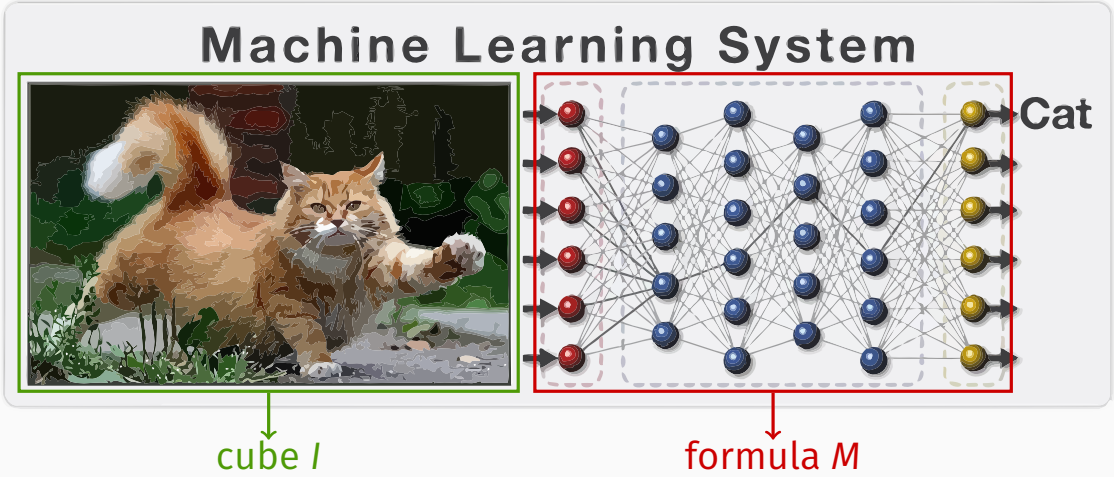

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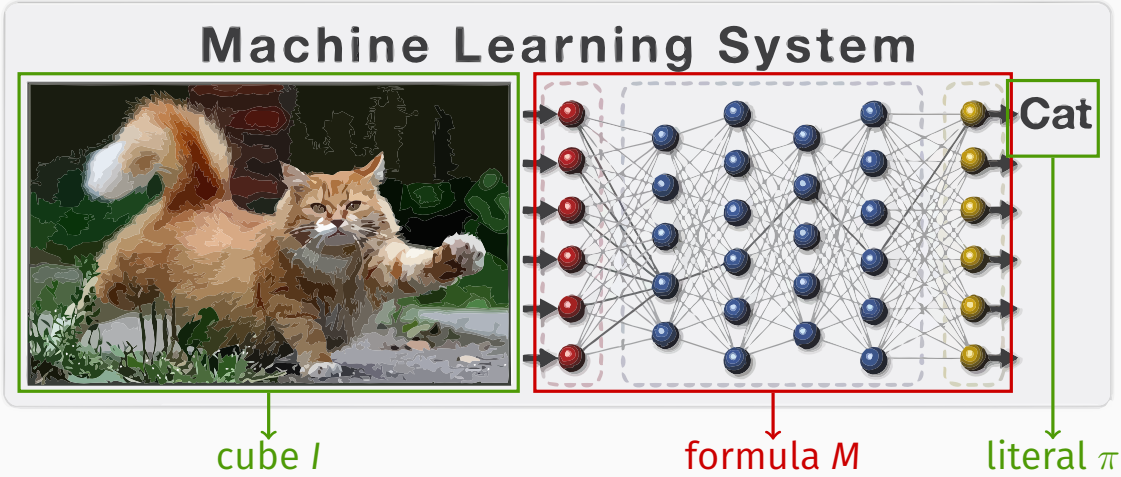

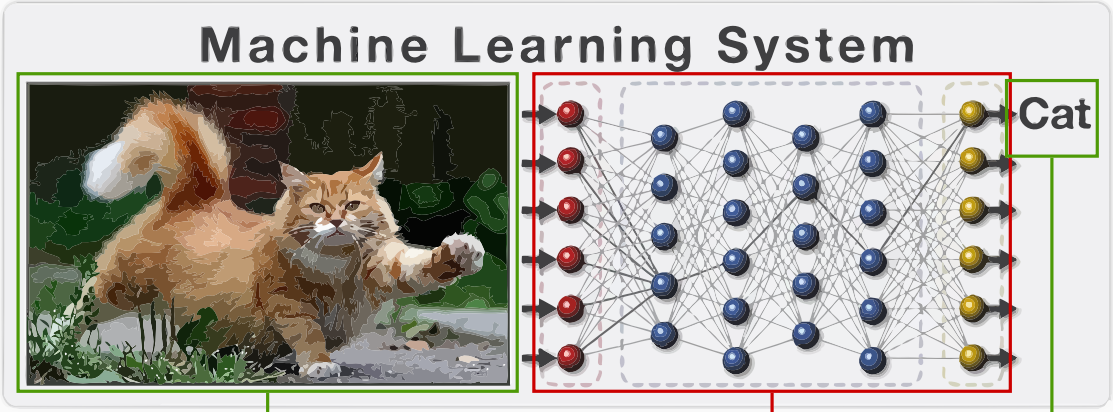

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Abductive explanations of ML models

[INMS19]

given a ***classifier*** M, a ***cube*** l and a ***prediction*** π,

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Abductive explanations of ML models

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$$E_m \wedge M \not\models \bot$$

and

$$E_m \wedge M \models \pi$$

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and

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iterative **explanation procedure**

1. $E_m \wedge M \not\models \bot$

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2. $E_m \wedge M \models \pi \iff E_m \models (M \rightarrow \pi)$

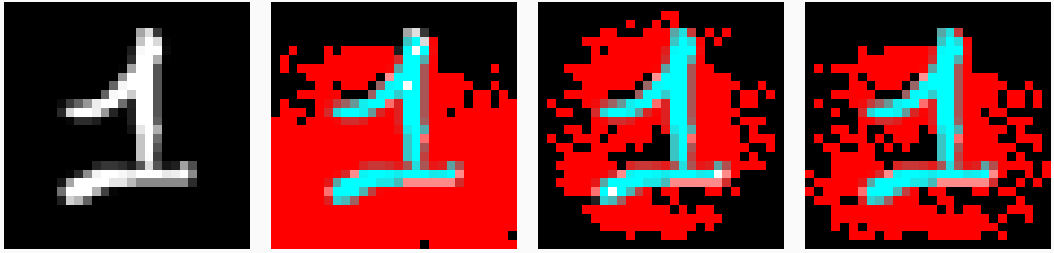
E_m is a ***prime implicant*** of $M \rightarrow \pi$

Computing one subset-minimal explanation

Input: model M, initial cube I, prediction π
Output: ***Subset-minimal*** explanation E_m

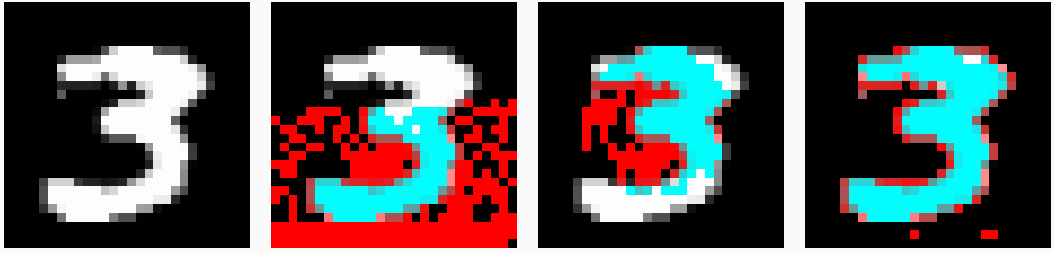
```
begin
  for l ∈ I :
    if Entails(I \ {l}, M → π) :          # make an oracle call
      I ← I \ {l}
  return I
end
```

Examples



(a) (b) (c) (d)

Figure 3: Possible minimal explanations for digit one.



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principled approach to XAI

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global explanations!

What next?

enumeration of **explanations**?

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preferences over explanations?

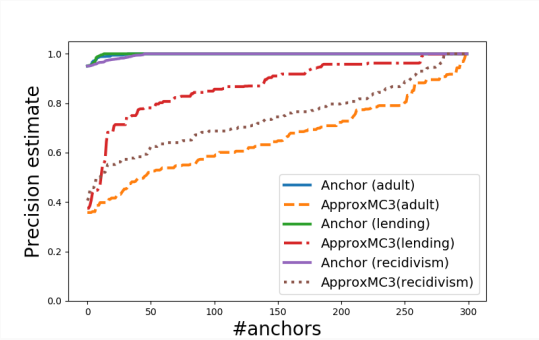
enumeration of **explanations?**

preferences over explanations?

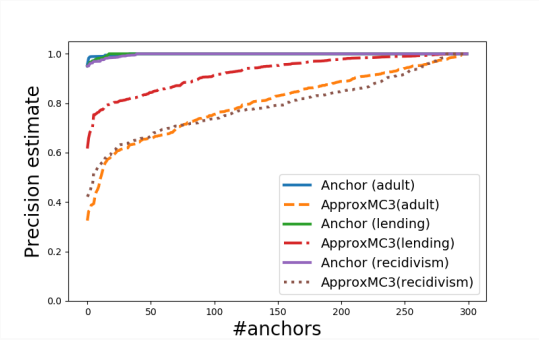
assessment heuristic approaches!

Assessing heuristic explanations

[NSM+19]



unconstrained feature space



samples with $\leq 50\%$ difference

PySAT and RC2

PySAT is a Python framework for
quick prototyping with SAT oracles

https://pysathq.github.io/

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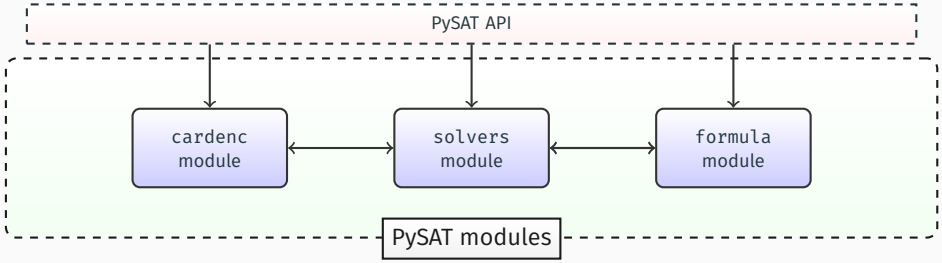
RC2 is a MaxSAT solver that
won two *complete* categories
of MaxSAT Evaluations 2018 & 2019

(FLoC Olympic Games 2018)

https://maxsat-evaluations.github.io/

Overview of PySAT

[IMM18]



Overview of PySAT

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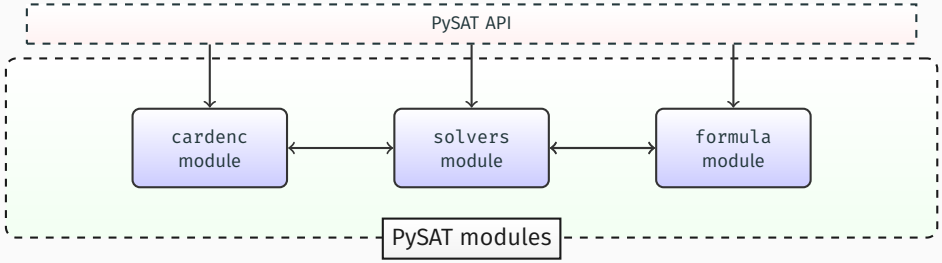

- Open source, available on github

Overview of PySAT

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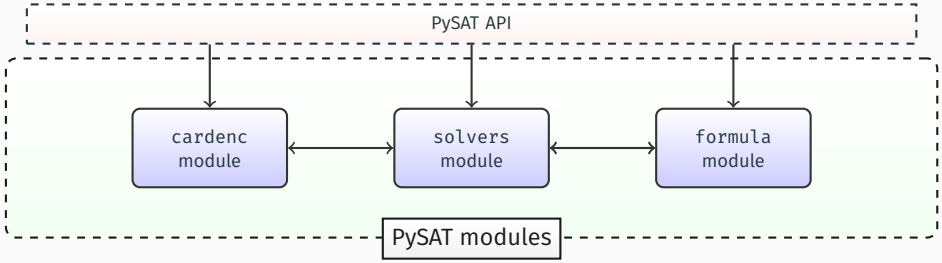

- Open source, available on github
- Comprehensive list of SAT solvers
- Comprehensive list of cardinality encodings
- Fairly comprehensive documentation
- Several use cases

Available solvers

Solver	Version
Glucose	3.0
Glucose	4.1
Lingeling	bbc-9230380-160707
Minicard	1.2
Minisat	2.2 release
Minisat	GitHub version
MapleCM	SAT competition 2018
Maplesat	MapleCOMSPS_LRB
...	...

- Solvers can either be used incrementally or non-incrementally
- Tools can use multiple solvers, e.g. for hitting set dualization or CEGAR-based QBF solving
- **URL**: `https://pysathq.github.io/docs/html/api/solvers.html`

Formula manipulation

Features
CNF & Weighted CNF (WCNF)
Read formulas from file/string
Write formulas to file
Append clauses to formula
Negate CNF formulas
Translate between CNF and WCNF
ID manager

- **URL**: `https://pysathq.github.io/docs/html/api/formula.html`

Available cardinality encodings

Name	Type
pairwise	AtMost1
bitwise	AtMost1
ladder	AtMost1
sequential counter	AtMost*k*
sorting network	AtMost*k*
cardinality network	AtMost*k*
totalizer	AtMost*k*
mtotalizer	AtMost*k*
kmtotalizer	AtMost*k*

- Also AtLeast*K* and Equals*K* constraints
- **URL**: `https://pysathq.github.io/docs/html/api/card.html`

Installation & info

- Installation:
  ```
  $ [sudo] pip2|pip3 install python-sat
  ```

- Website: https://pysathq.github.io/

Basic interface – Python3 shell

```
>>> from pysat.card import *
>>> am1 = CardEnc.atmost(lits=[1, -2, 3], encoding=EncType.pairwise)
>>> print(am1.clauses)
[[-1, 2], [-1, -3], [2, -3]]
>>>
>>> from pysat.solvers import Solver
>>> with Solver(name='m22', bootstrap_with=am1.clauses) as s:
...     if s.solve(assumptions=[1, 2, 3]) == False:
...         print(s.get_core())
[3, 1]
```

Basic interface – Python3 script

```
#!/usr/local/bin/python3
from sys import argv

from pysat.formula import CNF
from pysat.solvers import Glucose3, Solver

formula = CNF()
formula.append([-1, 2, 4])
formula.append([1, -2, 5])
formula.append([-1, -2, 6])
formula.append([1, 2, 7])

g = Glucose3(bootstrap_with=formula.clauses)

if g.solve(assumptions=[-4, -5, -6, -7]) == False:
    print("Core: ", g.get_core())
```

Example: naive (deletion) MUS extraction

Input : Set $\mathcal{F}$
Output: Minimal subset $\mathcal{M}$
begin
  $\mathcal{M} \leftarrow \mathcal{F}$
  foreach $c \in \mathcal{M}$ **do**
    if $\neg\mathsf{SAT}(\mathcal{M} \setminus \{c\})$ **then**
      $\mathcal{M} \leftarrow \mathcal{M} \setminus \{c\}$ // If $\neg\mathsf{SAT}(\mathcal{M} \setminus \{c\})$, then $c \notin$ MUS
  return $\mathcal{M}$ // Final $\mathcal{M}$ is MUS
end

- Number of predicate tests: $\mathcal{O}(m)$

[CD91, BDTW93]

Naive MUS extraction I

```
def main():
    cnf = CNF(from_file=argv[1])   # create a CNF object from file
    (rnv, assumps) = add_assumps(cnf)

    oracle = Solver(name='g3', bootstrap_with=cnf.clauses)

    mus = find_mus(assumps, oracle)
    mus = [ref - rnv for ref in mus]
    print("MUS: ", mus)

if __name__== "__main__":
  main()
```

Naive MUS extraction II

```
def add_assumps(cnf):
    rnv = topv = cnf.nv
    assumps = []                        # list of assumptions to use
    for i in range(len(cnf.clauses)):
        topv += 1
        assumps.append(topv)            # register literal
        cnf.clauses[i].append(-topv)    # extend clause with literal
    cnf.nv = cnf.nv + len(assumps)      # update # of vars
    return rnv, assumps

def main():
    cnf = CNF(from_file=argv[1])   # create a CNF object from file
    (rnv, assumps) = add_assumps(cnf)

    oracle = Solver(name='g3', bootstrap_with=cnf.clauses)

    mus = find_mus(assumps, oracle)
    mus = [ref - rnv for ref in mus]
    print("MUS: ", mus)

if __name__== "__main__":
  main()
```

Naive MUS extraction III

```
from sys import argv

from pysat.formula import CNF
from pysat.solvers import Solver


def find_mus(assmp, oracle):
    i = 0
    while i < len(assmp):
        ts = assmp[:i] + assmp[(i+1):]
        if not oracle.solve(assumptions=ts):
            assmp = ts
        else:
            i += 1
    return assmp
```

Naive MUS extraction III

```
from sys import argv

from pysat.formula import CNF
from pysat.solvers import Solver


def find_mus(assmp, oracle):
    i = 0
    while i < len(assmp):
        ts = assmp[:i] + assmp[(i+1):]
        if not oracle.solve(assumptions=ts):
            assmp = ts
        else:
            i += 1
    return assmp
```

Demo

A less naive MUS extractor

```
def clset_refine(assmp, oracle):
    assmp = sorted(assmp)
    while True:
        oracle.solve(assumptions=assmp)
        ts = sorted(oracle.get_core())
        if ts == assmp:
            break
        assmp = ts
    return assmp
# ...
def main():
    cnf = CNF(from_file=argv[1])   # create a CNF object from file
    (rnv, assumps) = add_assumps(cnf)

    oracle = Solver(name='g3', bootstrap_with=cnf.clauses)

    assumps = clset_refine(assumps, oracle)
    mus = find_mus(assumps, oracle)
    mus = [ref - rnv for ref in mus]
    print("MUS: ", mus)

if __name__== "__main__":
  main()
```

[Sur12]

given K **agents** and K **goals** on a map of **size** $N \times M$,

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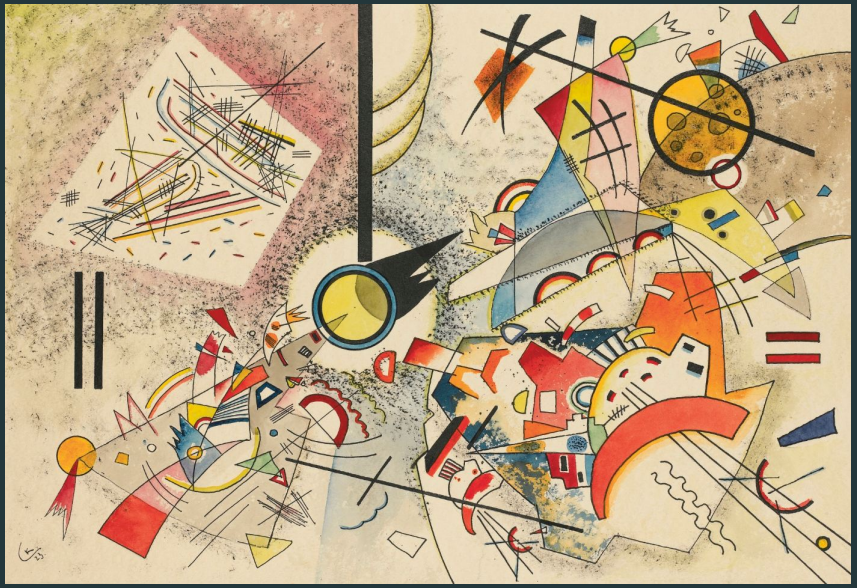
demo for MAPF@Minecraft+PySAT:

`https://reason.di.fc.ul.pt/~aign/storage/mcmapf2.mov`

6 SAT Oracles & Proof Complexity



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- Some basic problems, like pigeon-hole principle, cannot have short Resolution Refutations
- From proof complexity point of view, Resolution is regarded as a rather weak proof system
- Recent efforts for developing efficient implementations of stronger proof systems:
  - Extended Resolution (ExtRes)
  - DRAT
  - Cutting Planes (CP)
  - Dual-Rail Maximum Satisfiability (DRMaxSAT)

Dual-Rail Maximum Satisfiability (DRMaxSAT)

DRMaxSAT - General Idea

- Translates a CNF formula $\mathcal{F}$ using the Dual-Rail Encoding
- Uses a MaxSAT algorithm to obtain the cost of the encoded formula
- Determines the satisfiability of $\mathcal{F}$ based on the cost of the encoded formula

Dual-Rail Encoding (DRE)

[DAC87, AI99]

Input: $\mathcal{F}$ CNF formula with N variables $X = \{x_1, \ldots, x_N\}$

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Output: MaxSAT problem $< \mathcal{H}, \mathcal{S} >$:

- for each $x_i \in X$:
  - associate new variables p_i and n_i

$$x_i = 1 \text{ iff } p_i = 1, \text{ and } x_i = 0 \text{ iff } n_i = 1$$

  - add to $\mathcal{S}$ the clauses (p_i) and (n_i)
  - add to $\mathcal{H}$ the clause $(\neg p_i \vee \neg n_i)$ ($\mathcal{P}$ clauses)

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  - add to $\mathcal{S}$ the clauses (p_i) and (n_i)
  - add to $\mathcal{H}$ the clause $(\neg p_i \vee \neg n_i)$ ($\mathcal{P}$ clauses)

- for each clause $c \in \mathcal{F}$ add to $\mathcal{H}$ the clause c':
  - if $x_i \in c$ then $\neg n_i \in c'$
  - if $\neg x_i \in c$ then $\neg p_i \in c'$

DRMaxSAT: DRE Example

$$\mathcal{F} = \{(\neg x_1 \vee \neg x_2), (x_1), (x_2), (\neg x_2)\}$$

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- MaxSAT problem $< \mathcal{H}, \mathcal{S} >$
- for x_1:
  - create p_1 and n_1
  - add (p_1), (n_1) to $\mathcal{S}$
  - add $(\neg p_1 \vee \neg n_1)$ to $\mathcal{H}$
- for x_2:
  - create p_2 and n_2
  - add (p_2), (n_2) to $\mathcal{S}$
  - add $(\neg p_2 \vee \neg n_2)$ to $\mathcal{H}$

DRMaxSAT: DRE Example

$$\mathcal{F} = \{(\neg x_1 \vee \neg x_2), (x_1), (x_2), (\neg x_2)\}$$

- MaxSAT problem $< \mathcal{H}, \mathcal{S} >$
- for $(\neg x_1 \vee \neg x_2)$:
  - add $(\neg p_1 \vee \neg p_2)$ to $\mathcal{H}$
- for (x_1), (x_2), $(\neg x_2)$:
  - add $(\neg n_1)$, $(\neg n_2)$, $(\neg p_2)$ to $\mathcal{H}$

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$\mathcal{F} = \{(\neg x_1 \vee \neg x_2), (x_1), (x_2), (\neg x_2)\}$

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MaxSAT Cost: 3

DRMaxSAT: DRE Theorem

Theorem
$\mathcal{F}$ is satisfiable iff there is a truth assignment satisfying $\mathcal{H}$ that satisfies at least N clauses in $\mathcal{S}$. [SAT17]

DRMaxSAT: DRE Theorem

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Example: $N = 2$ and MaxSAT cost 3, thus $\mathcal{F}$ is unsatisfiable.

DRMaxSAT: General Pseudo-Code

input: $\mathcal{F}$

$\mathsf{HEnc}(\mathcal{F}) = \langle \mathcal{H}, \mathcal{S} \rangle \leftarrow \mathsf{DualRailEncode}(\mathcal{F})$

cost $\leftarrow \mathsf{ApplyMaxSAT}(\mathsf{HEnc}(\mathcal{F}))$

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DRMaxSAT: Theoretical Results

What is the power of DRMaxSAT?

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What is the power of DRMaxSAT?

Considered three MaxSAT approaches with DRMaxSAT:

- MaxSAT Resolution
- Core-guided MaxSAT Algorithms
- Implict Hitting Set MaxSAT Algorithms (MaxHS-like algorithms)

DRMaxSAT: Summary of Theoretical Results on MaxSAT Resolution

Theorem
Multiple DRMaxSAT simulates tree-like Resolution. [Source: AAAI18]

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Weighted DRMaxSAT simulates general Resolution. [Source: AAAI18]

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- DRMaxSAT is strictly stronger than Resolution
- DRMaxSAT does not simulate Cutting Planes

DRMaxSAT: Summary of Theoretical Results with Core-guided Algorithms

Theorem
Core-guided MaxSAT with the dual-rail encoding p-simulates unrestricted Resolution.

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[Source: SAT17,AAAI18]

- Core-guided MaxSAT with the dual-rail encoding is strictly stronger than Resolution

DRMaxSAT: Summary of Theoretical Results with MaxHS-like Algorithms

Theorem
MaxHS-like MaxSAT Algorithms with the dual-rail encoding refutes both PHP and 2PHP in polynomial time. *[Source: SAT19]*

DRMaxSAT: Experimental Results

What is the behaviour of DRMaxSAT in practice?

DRMaxSAT: Experimental Results

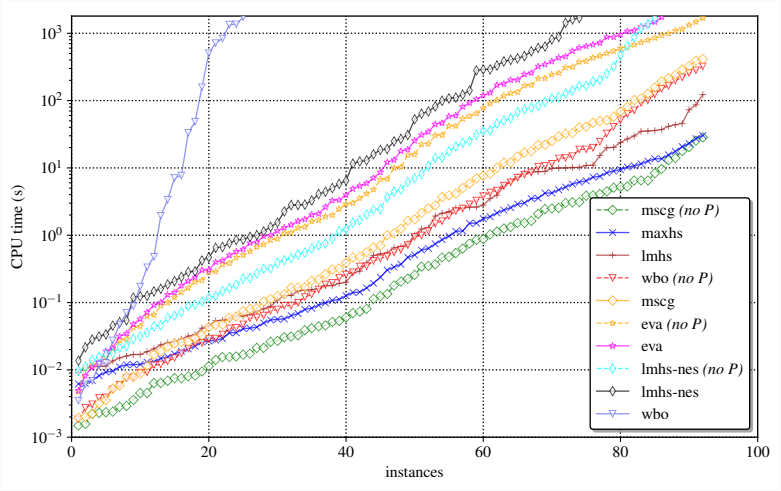
What is the behaviour of DRMaxSAT in practice?

Several problems/principles hard for resolution:

- (PHP) Pigeonhole Principle
- (2PHP) Doubled Pigeonhole Principle
- (MCB) Mutilated Chessboard Principle
- (URQ) Urquhart/Tseitin formulas
- (COMB) Combination formulas $\text{PHP}_m^{m+1} \vee \text{URQ}_{n,i}$

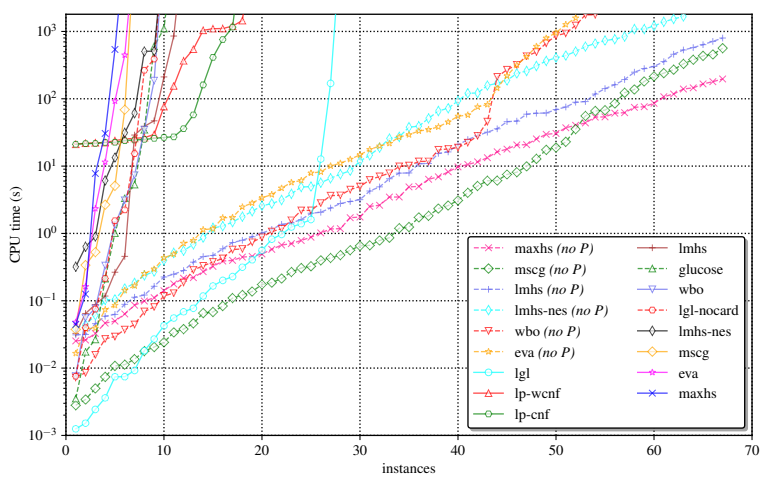
DRMaxSAT: Experimental Results - PHP

[Source: SAT17]



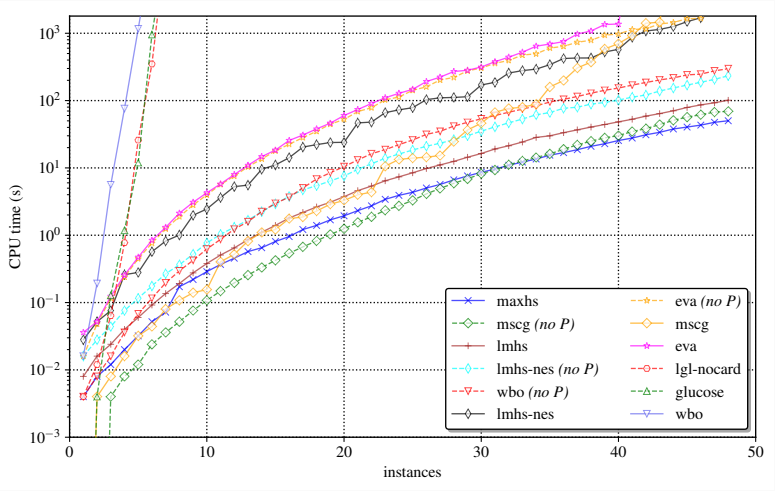
DRMaxSAT: Experimental Results - 2PHP

[Source: AAAI18]



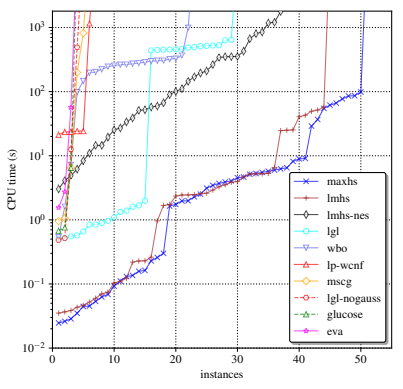
DRMaxSAT: Experimental Results - MCB

[Source: AAAI18]

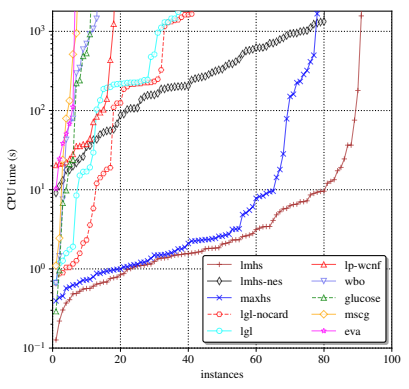


DRMaxSAT: Experimental Results - URQ & COMB

[Source: SAT17]



URQ



COMB

7 A Glimpse of the Future

Take home messages

- Remarkable improvements in (CDCL) SAT solver performance
  - SAT is a low-level, but very powerful problem solving paradigm

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  - SAT is a low-level, but very powerful problem solving paradigm
- Wide range of applications of SAT oracles
  - High profile applications, e.g. verification of ML models & XAI
  - Solving problems (well) beyond NP
  - Recent inroads in proof complexity
    - Ongoing arms race for proof systems stronger than resolution/clause learning, e.g. extended resolution, cutting planes and DRMaxSAT

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    - Ongoing arms race for proof systems stronger than resolution/clause learning, e.g. extended resolution, cutting planes and DRMaxSAT
- Also, there are other success stories, e.g. ILP

Some final notes

- There is an ongoing revolution on problem solving with SAT oracles
  - E.g. QBF, model-based diagnosis, explainability, theorem proving, program synthesis, XAI, ...
  - But also with ILP & SMT oracles
- The use of SAT/ILP/SMT oracles is impacting problem solving for many different complexity classes
  - With well-known representative problems, e.g. QBF, #SAT, etc.
  - More to be expected
    - E.g. the age of **modular reasoning**?

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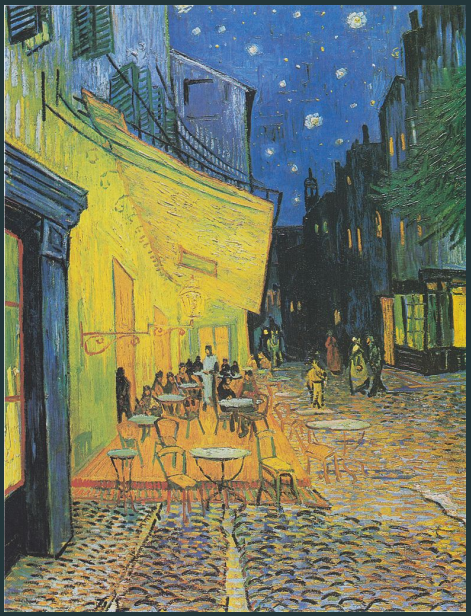
- The use of SAT/ILP/SMT oracles is impacting problem solving for many different complexity classes
  - With well-known representative problems, e.g. QBF, #SAT, etc.
  - More to be expected
    - E.g. the age of **modular reasoning**?
- Many fascinating oracle-related research topics out there **!**
  - Additional connections with rigorous reasoning in ML expected

Sample of tools

- PySAT
- SAT solvers:
  - MiniSat
  - Glucose
- MaxSAT solvers:
  - RC2
  - MSCG
  - OpenWBO
  - MaxHS
- MUS extractors:
  - MUSer
- MCS extractors:
  - mcsXL
  - LBX
  - MCSls
- Many other tools available from the ReasonLab server

Questions?



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