On Reducing Maximum Independent Set to Minimum Satisfiability

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Problems statements

MIS: compute the **largest** number of pairwise non-connected vertices in $\mathcal{G}$.
MVC: compute the **smallest** number of vertices in $\mathcal{G}$ that are incident to all edges of $\mathcal{G}$.

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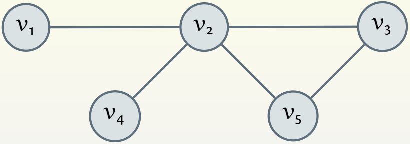
Given $\mathcal{F}$, $\mathcal{M} \subseteq \mathcal{F}$ is a MaxFalse solution $\Leftrightarrow$ $\mathcal{F} \setminus \mathcal{M}$ is a MinSAT solution.

MIS and MaxFalse

MIS $\leftrightarrow$ **MaxFalse**

MVC $\leftrightarrow$ **MinSAT**

Basic reduction



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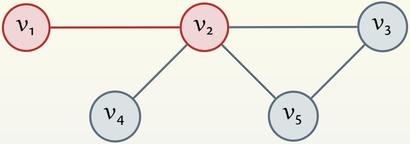

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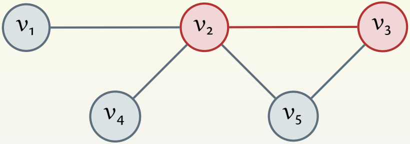

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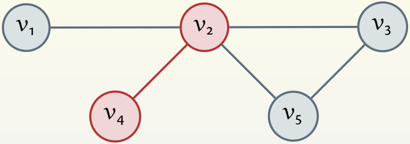

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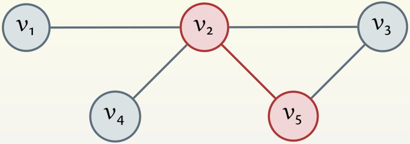

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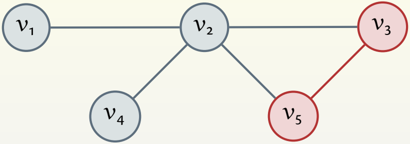

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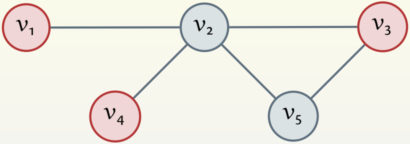

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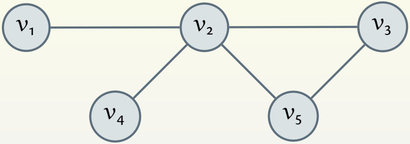
Given a graph $\mathcal{G} = (V, E)$, **basic reduction** constructs a formula $\mathcal{F}$ with exactly $|V|$ clauses and $|E|$ variables.

Greedy reduction



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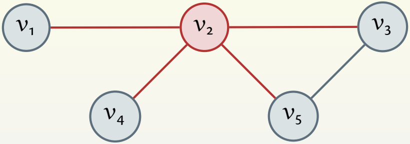

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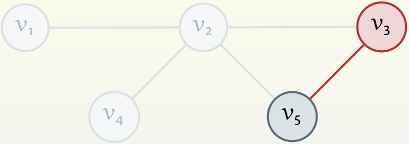

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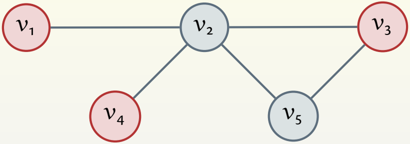

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Given a graph $\mathcal{G} = (V, E)$, **greedy reduction** constructs a formula $\mathcal{F}$ with exactly $|V|$ clauses and $\leqslant |V|$ variables.

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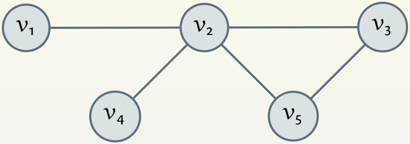
  2. no new connection

     given two clauses $\neg x_1 \vee x_2$ and $\neg x_3$, variables x_2 and x_3 are not compatible

Variable compatibility[1]



(a) Graph $\mathcal{G}$

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\begin{aligned}
c_1 &= x_1 \\
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(b) Set of clauses for $\mathcal{G}$

	x_1	x_2	x_3	x_4	x_5
x_1	–				
x_2	$*_2$ $*_{1,3}$	–			
x_3	$*_2$ $*_{1,4}$		–		
x_4	$*_2$ $*_{1,5}$			–	
x_5	$*_{1,5}$	$*_3$	$*_{3,4}$		–

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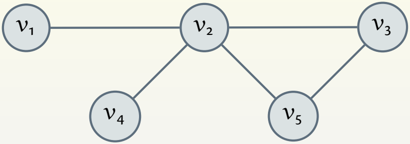

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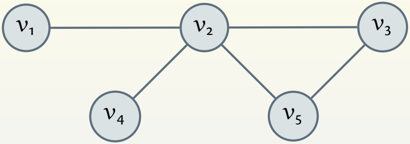

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Weighting clauses and removing duplicates

		basic			greedy			greedy+vc		
		vars	clauses	time[2]	vars	clauses	time[2]	vars	clauses	time[2]
Instance	c-fat200-1	18366	200	0.4	188	200	0.05	35	37	0
	c-fat200-2	16665	200	0.75	176	200	0.07	16	18	0
	c-fat200-5	11427	200	0.96	142	200	0.07	5	7	0
	c-fat500-1	120291	500	—	486	500	0.63	78	80	0
	c-fat500-10	78123	500	—	374	500	0.53	6	8	0
	c-fat500-2	115611	500	—	474	500	0.51	38	40	0
	c-fat500-5	101559	500	—	436	500	0.37	14	16	0

[2] Running time for MinSatz.

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Experimental results

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Native	✓								✓
MaxFalse-based		✓		✓		✓		✓	✓
Direct MaxSAT			✓		✓		✓		✓

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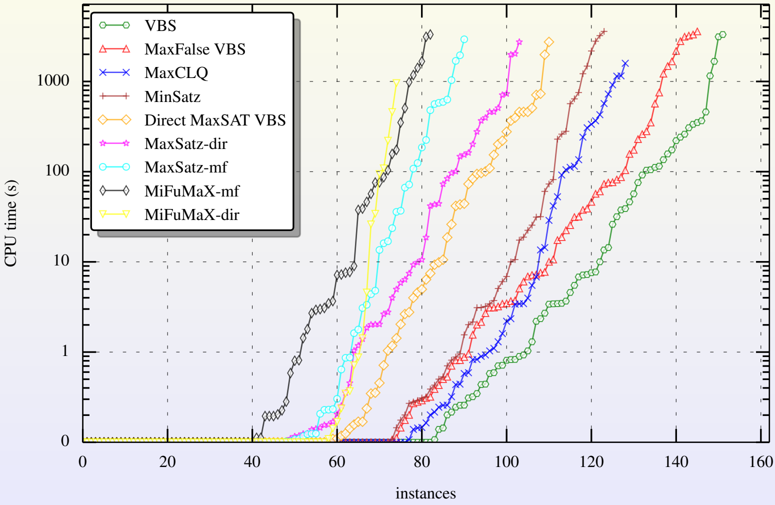
Crafted Clq	**66**	59	43	36	19	30	45	74	76
BCP	63	**65**	61	55	56	53	66	72	76
Total	**129**	124	104	91	75	83	111	146	152

Performance of the considered approaches



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2. portfolios of solvers for NP-hard graph problems
3. development of efficient MinSAT solvers

Thank you for your attention!