Quantified Maximum Satisfiability: A Core-Guided Approach

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Applications — optimization problems with quantified constraints.

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Example

$$\xi = \exists e_1, e_2\, \forall x_1, x_2.\ (\neg e_1 \wedge \neg e_2) \rightarrow (x_1 \vee x_2)$$

Quantified MaxSAT

$$\psi = \exists E\, \overrightarrow{Q}.\ \varphi$$

Definition

Assignment $\mathcal{A}_E$ is a ***solution*** of ψ iff $\overrightarrow{Q}.\ \varphi\big|_{\mathcal{A}_E}$ is true.

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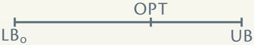

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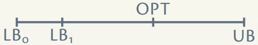

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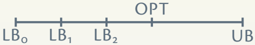

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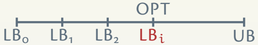

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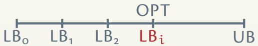

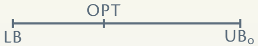

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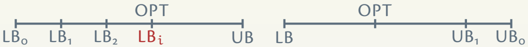

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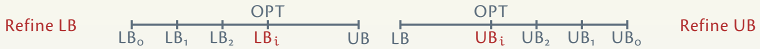

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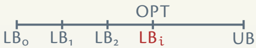

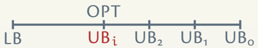

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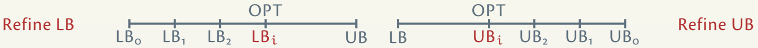

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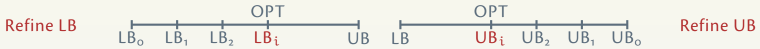

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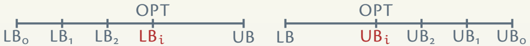

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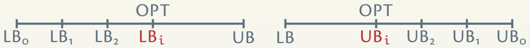

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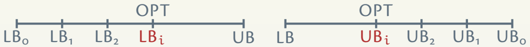

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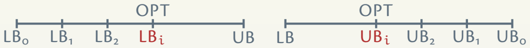

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Formula $\varphi_C = \varphi \wedge \varphi'_S$, $\varphi'_S \subseteq \varphi_S$, is an ***unsatisfiable core*** of ψ' if $\exists E\ \overrightarrow{Q}.\ \varphi_C$ is false.

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  $(\mathrm{st}, \varphi_C, \mathcal{A}_E) \leftarrow \mathtt{QBF}(\psi'_R)$ # calling a QBF oracle

  if st $=$ true:
    return $\mathcal{A}_E$

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  foreach $c \in \mathtt{Soft}(\varphi_C)$: # relaxing the core
    let r be a new relaxation variable
    $R \leftarrow R \cup \{r\}$
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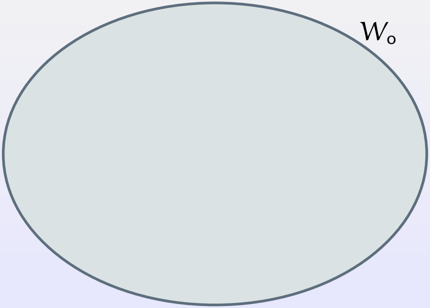
Gradual strengthening of abstractions until a solution is found



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Full expansion: $\exists X.\ \bigwedge_{\nu \in \{0,1\}^{|Y|}} (\varphi_H \wedge \varphi_S)\big|_\nu$

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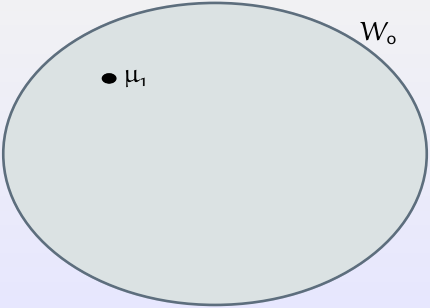
Gradual strengthening of abstractions until a solution is found



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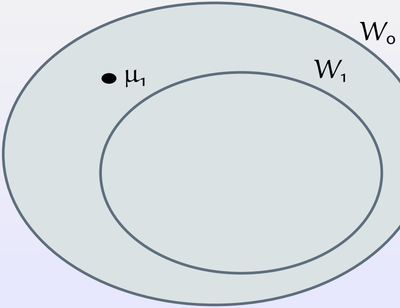

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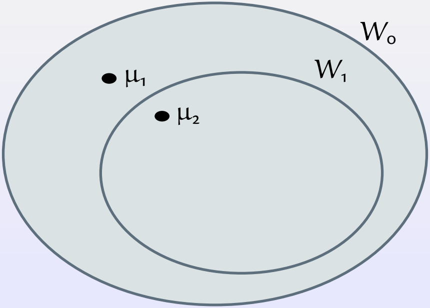

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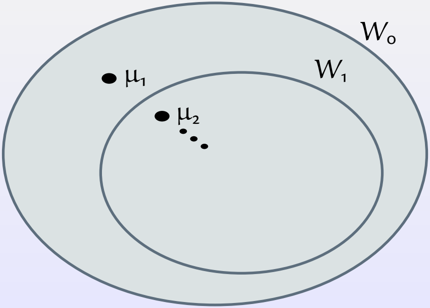

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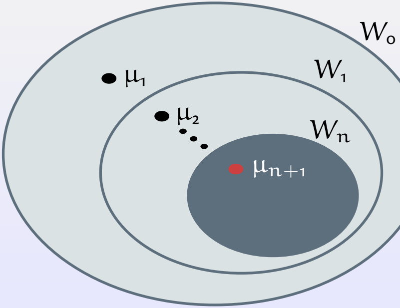

Computing Cores in CEGAR-based 2QBF

input : $\exists X \forall Y.\ \varphi_H \wedge \varphi_S$

$\omega \leftarrow \emptyset$

while true:

$\quad \varphi \leftarrow \mathtt{CNF}(\bigwedge_{\nu \in \omega} \varphi_H|_\nu) \cup \bigwedge_{\nu \in \omega} \varphi_S|_\nu$

$\quad (st_1, \mu, \varphi_C) \leftarrow \mathtt{SAT}(\varphi)$ `# candidate`

$\quad$ **if** $st_1 =$ false:

$\quad\quad \varphi'_S \leftarrow \{c \in \varphi_S \mid c' \in \varphi_C, \nu \in \omega, c' = c|_\nu\}$

$\quad\quad$ **return** (false, $\varphi_H \wedge \varphi'_S$) `# no candidate found`

$\quad (st_2, \nu) \leftarrow \mathtt{SAT}\left(\neg(\varphi_H \wedge \varphi_S)|_\mu\right)$ `# counterexample`

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Smallest MUS Problem

Definition

Formula ψ^*, $\psi^* \subseteq \varphi$, is called a ***smallest MUS*** of φ if

1. ψ^* is unsatisfiable
2. for any MUS ψ, $\psi \subseteq \varphi$, the following holds $|\psi^*| \leqslant |\psi|$

Example

$$\varphi = \{\ x_2 \vee \neg x_3 \vee \neg x_4,\ \ x_1 \vee x_2,\ \ x_3,\ \ \neg x_1,\ \ x_4,\ \ \neg x_2\ \}$$

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Example

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  - **find** $\mathcal{A}_S \in \mathcal{M}(\varphi_{unsat})$ s. t. $\forall \mathcal{B}_S \in \mathcal{M}(\varphi_{unsat})$: $f(\mathcal{A}_S) \leqslant f(\mathcal{B}_S)$

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  - $\exists S\, \forall X.\ \neg \varphi_R \wedge \varphi_S$, # core-guided search
    where $\varphi_S = \{\neg s_1, \dots, \neg s_m\}$ # soft constraints

Improvements of the Approach

Digger — state of the art for SMUS.

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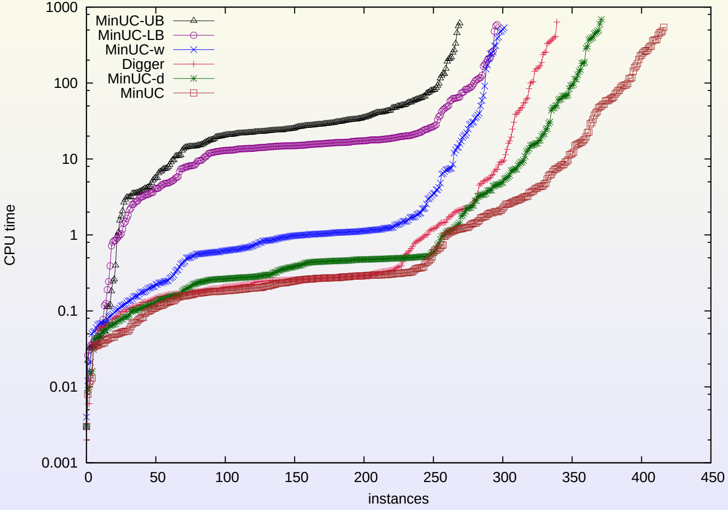
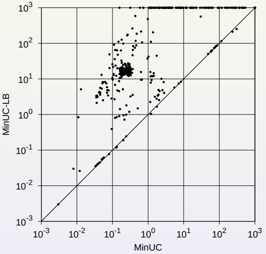
- **any** MCS is an unsatisfiable core of $\exists S\, \forall X.\ \neg\varphi_R \wedge \varphi_S$ `# see the paper`

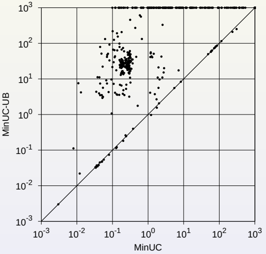
Performance Comparison: MinUC vs Digger



Performance Comparison: Linear Search vs Core-Guided

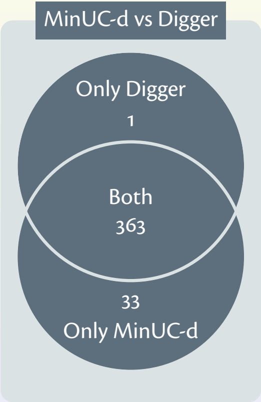


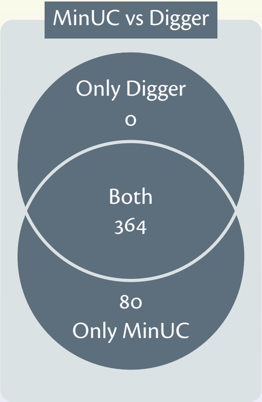
(a) MinUC vs MinUC-LB



(b) MinUC vs MinUC-UB

Number of Solved Instances





Total number of instances — 682.

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- *CEGAR-based vs DPLL-based* comparison (unsatisfiable cores)

Thank you for your attention!